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SUPPLEMENTARY MEETING

15 December, 1955

WILLIAM KELLY WALLACE, C.B.E., President, in the Chair

RANKINE MEMORIAL LECTURE

The President said that the lecture to be delivered tonight by Sir Richard Southwell was being held to commemorate the centenary of the appointment to the Chair of Civil Engineering and Mechanics of Glasgow University of William John Macquorn Rankine. The lecture had been first delivered to the University of Glasgow on Monday, 12 December, 1955, and was being repeated tonight to enable those engineers resident in the London area to pay their tribute to the memory of Professor Rankine.

William John Macquorn Rankine had been born in 1820 in Edinburgh and educated at Ayr Academy, the High School of Glasgow, and the University of Edinburgh. He had been a pupil of John Benjamin MacNeill, and had been engaged on river improvements, waterworks, harbour works, and railways in Ireland. He had become an Associate of the Institution of Civil Engineers, but resigned in 1857 on the establishment of the Institute of Engineers in Scotland, of which he had been elected the first President. He had been appointed to the Chair of Civil Engineering and Mechanics at Glasgow University on 7 November, 1855.

Sir Richard Vynne Southwell, M.A., LL.D., D.Sc., D.Eng., Hon.M.I.Mech.E., F.R.S., had been born in Norwich in 1888 and educated at Norwich School and Trinity College, Cambridge. He had been Superintendent of the Aerodynamics Department of the National Physical Laboratory, 1920-25; University Lecturer in Mathematics and Fellow of Trinity College, Cambridge, 1925-29; Professor of Engineering Science and Fellow of Brasenose College, Oxford, 1929-42; and Rector of the Imperial College of Science and Technology, 1942-48. Among his publications were "Relaxation methods in engineering science", 1940, and Papers on elasticity, theory of structures, hydrodynamics, etc., in the Transactions of the Royal Society and the Philosophical Magazine. He had been awarded the Ewing Medal by the

Institution in 1946. He had delivered the James Forrest Lecture in 1930 and in 1948—and there were very few lecturers who had given the James Forrest Lecture twice.

Sir Richard Southwell then delivered his Lecture.

W. J. M. RANKINE : A COMMEMORATIVE LECTURE †

delivered on 12 December, 1955, in Glasgow,

by Sir Richard V. Southwell, M.A., LL.D., D.Sc., D.Eng.,
Hon.M.I.Mech.E., F.R.S.

FOREWORD

In a lecture intended for a general audience, and to be delivered within one hour, there could be no question of treating systematically all of Rankine's many scientific papers. I thought it best to concentrate on some of those which I believe to have had most lasting influence, and in the printed lecture to relegate to appendices matters not suited to an oral presentation, along with bibliographical references.

1. On 10 December, 1855, William John Macquorn Rankine, appointed to your Regius Chair of Civil Engineering and Mechanics in a document read to the Senate one week earlier, fulfilled the requirement which the Senate had then imposed upon him by reading before them a Latin dissertation 'On the Harmony of Theory and Practice in Mechanics.' (Your library contains it, and a translation into English. Its substance occupies the first 11 pages of his *Applied Mechanics*.) Tonight we are met to commemorate that event—as exactly as the preoccupations of a busy term allow. It marked the start of seventeen years of intense activity—remarkable even in those times and in this home of giants, where James Thomson, his successor, was to serve for a nearly equal term and his brother William to occupy for more than 50 years your Chair of Natural Philosophy: a period in which books on Applied Mechanics, Civil Engineering, Shipbuilding, Steam-Engines and other Prime Movers, Machinery and Mill-Work, flowed from his pen along with articles for encyclopaedias and engineering journals, yet left him able, somehow, to find time for original papers numbering more than 125.¹

His was a time of world-famous engineers; yet Tait could truthfully write of him in his Memoir:² "Few, if any, practical engineers have contributed so much to abstract science, and in no case has scientific study been applied with more effect to practical engineering."

2. It is a privilege to have part in your celebrations; an honour altogether undeserved, to have been entrusted with the delivery of this address. An honour, but an ordeal! (So much so that, had your invitation not come many months ago, I think I would not have dared to accept a task for which I am so patently ill-equipped. But Professor Small, understanding human nature, sent it at a date far enough

† The lecture was first delivered on 12 December, 1955, before the University of Glasgow, to commemorate the centenary of Rankine's appointment to the Queen Victoria Chair of Civil Engineering and Mechanics; and is published by courtesy of the University of Glasgow, in the written form of which that spoken lecture was an abbreviation.

ahead to blur my vision of dangers now only too plainly manifest.) It has been said that a lecture should include some things known both to the lecturer and to his audience; some things known, before its delivery, only to the lecturer; and some that neither he nor his audience really comprehends. The first requirement, by assiduous 'homework', I might hope to satisfy—and the third, so far as lack of comprehension is required of me; but of Rankine, who spent most of his working life in Glasgow, I could hope to tell you nothing of which you are not apprised, and I know that despite my reading I am ignorant of many things which you regard as common knowledge.

I confess that when I started my knowledge of Rankine's work was very sketchy. At Cambridge, in my far-off undergraduate years, I had met the 'Rankine cycle' and the 'Rankine-Gordon strut formula', and had been told—but only incidentally—of his theory of earth-pressure and the stability of retaining walls. But few of us then, I think, had curiosity about the men who had discovered the things we learned of, and since that time my interests have lain in other fields; so it was not till much later that I learned to see him as a really great Victorian, recognized both here and abroad as meriting a place in the pantheon of 19th-century science. (In 1950, at the Polytechnic Institute of Virginia, I found his statue dominating the hall of their Department of Engineering.) I came to know that his theory of earth-pressure blazed a trail which many have followed since (among them C. F. Jenkin, my predecessor at Oxford), so that now Soil Mechanics is a branch of study standing on its own. And in the war years, in the Civil Defence Committee, I came to realize that the 'Rankine-Hugoniot theory' is basic to a comprehension of 'blast waves'—a phenomenon then brought forcibly to notice. Here were, already, two major contributions in fields that I had not made my own; work in a third field (Thermodynamics) which I knew I should have to study carefully; and all that great mass of work which Rankine did as a teacher, by lectures and by textbooks. I could have no doubt, even when your invitation came, that acceptance of it would mean, for me, much reading!

3. For my task, as I envisaged it, would entail an attempt to assess Rankine's place in the history of science: must, if this celebration were to be more than a routine marking of an anniversary. A centenary is not the occasion for a mere *éloge*; nor would Rankine (if I picture him aright) have wished any claim to be made for him that is not made sincerely. We know that he has a place among the giants of his time. But what is that place exactly? Where does he stand in relation to his contemporaries? These were questions I saw that I must try to answer—knowing my answers would be, inevitably, coloured by my own interests, biassed by my own thinking, yet based, in all but one of the fields he made his own, on the judgement of others.

Because Rankine's was not, as ours is, a time of specialization—of men "knowing more and more about less and less". Never again would one man have the range of Leonardo, but one man could make advances in many fields—and Rankine did. Think of it! In Thermodynamics he and Clausius made like advances almost simultaneously;³ but his paper 'On the Stability of Loose Earth' appeared within 2 years of his installation—16 years before the work of Boussinesq;⁴ he, in his study of shock waves, was the pioneer whom Hugoniot followed 17 years later;⁵ and his work on wave-resistance and the lines of ships⁶ laid foundations for subsequent building by W. and R. E. Froude. Think, I say, of these things—and be indulgent of my survey!

4. I had first to gain an impression of the man himself, and for this I went to the Memoir by P. G. Tait⁷ which prefaces Millar's volume *Miscellaneous Scientific Papers*

a selection). It presents Rankine as seen by a contemporary, and having been revised some 8 years after his death (though written at the time) it need not be suspected of the tendency to over-praise sometimes detectable in more immediate obituaries. Moreover, Tait wrote from "actual acquaintance with Rankine's purely scientific work", though he "found it necessary to apply for assistance while attempting to discuss the merit of his more practical investigations". Thus his memoir gives us what I think we want tonight—a close-range view.

It has the qualities we term Victorian: complete trust in the inevitability of progress, and a high sense (almost amounting to complacency) of the contribution that men of science were making to the world's advance towards ultimate perfection; a leisurely style that presumes no scarcity of printers or of paper, and readers without our modern craving for compression; readiness for occasional relaxation, but not so far as to risk loss of dignity—like the cheerfulness of Dr. Johnson's college-friend who in his time had "tried to be a philosopher", dignity, ". . . I don't know how, —was always breaking in".⁸ Thus, of Rankine:—" . . . he could turn without apparent mental effort from the prattle of young children to the most formidable statement of new results in mathematical or physical science."

But the memoir really is evidence of these qualities in Tait rather than Rankine—whose "appearance was striking and prepossessing in the extreme" and whose "courtesy resembled almost that of a gentleman of the old school". Quite plainly, he was popular: when the 'Red Lions' of the British Association met in Edinburgh (in 1871) he "was hailed with universal acclaim as the Lion-King". In such circles he would sing, to his own accompaniment, songs that were published posthumously.⁹ Tait prints the whole of one, 'The Three-Foot-Rule', from which (expecting all of you to know it) I quote only the verse that has the swing of 'The British Grenadier':—

"Some talk of millimètres, and some of kilogrammes,
And some of décilitres, to measure beer and drams,
But I'm a British workman, too old to go to school,
So by pounds I'll eat, and by quarts I'll drink,
and I'll work by my three-foot rule."

5. But now I must turn to the question I raised earlier, of Rankine's place in relation to contemporary men of science. It has a present-day answer of a kind, and one that perhaps is worth citing, though in no way conclusive: I mean, the space allotted him in the '*Concise D.N.B.*', where the longer memoirs of the full-length *Dictionary*—themselves dependent on another variable, the differing styles of writing of their authors—are pruned to lengths which afford fair indication of judgements made by an authoritative board of editors. They depend as well, of course, on the relative importance which the editors attach to different callings, and as between a man of science and a poet or statesman I would not suggest that they are comparable. But here are some that do seem comparable, and I give them for what they are worth:¹⁰

J. W. M. Rankine	12 lines in ' <i>Concise D.N.B.</i> '
W. Thomson (Lord Kelvin)	81 " " "
P. G. Tait	33 " " "
G. G. Stokes	29 " " "
J. W. Strutt (Lord Rayleigh)	23 " " "
John Hopkinson	18 " " "

Isambard Brunel	12 lines in 'Concise D.N.B.'
Sylvanus Thompson	6 „ „ „
James Thomson	5 „ „ „
W. C. Unwin and John Perry	0 „ „ „

(I add, as pointing the need to compare like callings, that Browning and Tennyson get about 50 lines each, Jack Ketch (the executioner—not a contemporary, of course) gets 5 lines.)

6. Within a restricted field—Strength of Materials—Timoshenko's *History* provides a more authoritative estimate. Summarizing advances made in the period 1833–67, it devotes: 3 pages to each of William Fairbairn and Eaton Hodgkinson (Professor in University College, London, 1847), about 6 pages to each of Rankine and Bresse, Culmann, Wohler and Jourawski (1821–91), but *much* more (nearly 19 pages—most of them under Theory of Elasticity) to Barré de Saint-Venant.

Rankine's name first appears in its account of Railway Engineering, and in relation to the problem of fatigue. It cites the second¹⁰ of his papers (published in 1843) as "perhaps the first English paper in this field". His tests contradicted the theory then commonly accepted, that after axles have been run for several years the fibrous texture of their iron deteriorates, gradually becoming 'crystallised'; and he proposed "to form the journals with a large curve in the shoulder, *before going to the lathe* [*italics mine*], so that the fibre shall be continuous throughout". Much discussion followed, leading to no agreement; Robert Stephenson urged engineers to "pause before you arrive at the conclusion that iron is a substance liable to . . . molecular change from vibration . . ."; and one P. R. Hodge made a suggestion which Stephenson followed up: "to call in the aid of the microscope".

7. Timoshenko praises highly Rankine's theory of retaining walls (earth pressures), also his application to frames and arches of his 'method of parallel projections'; but sees the Rankine-Gordon formula, I think, as no more than a straightforward development of notions mainly due to Lamarle and Hodgkinson and Tredgold. I confess to a like opinion—and indeed, in all Rankine's work in the Theory of Structures, I can find no really outstanding contribution to Elasticity: none that compares, for example, with that which Saint-Venant had made two years before the occasion that we commemorate.¹¹ I admit to having an admiration for Saint-Venant that comes very near to reverence: his *Mémoire* marks, for me, the birth of a Theory of Elasticity really having application to engineering. But I do not think it inclines me to be unfair to other authors, and my view of the long and intricate papers that Rankine wrote in 1850, 1851 and 1855¹² on the Laws of Elasticity and their derivation from his 'Centrifugal Theory' (Molecular Vortex Hypothesis) does not differ greatly from that which Karl Pearson bases on vastly wider knowledge. (Timoshenko—being concerned with Strength of Materials—makes no comment on them in his *History*. Pearson reviewed them in one of the chapters he added to Todhunter's.¹³) He praises the service rendered by Rankine "in the sphere of terminology . . . by his introduction or precise definition of . . . useful names . . ." ('Stress' and 'strain', as now used technically, are his inventions; also 'potential energy'—which he defended against Herschel's criticisms in a paper read before your Philosophical Society in 1867)¹⁴; and he notes Rankine's priority, "which was fully admitted by Saint-Venant", in deriving expressions for transformation of the elastic coefficients. But he does not hesitate to pronounce some of his postulates insufficient, some of his theorems erroneous, some of his reasoning "rather vague"; and observes that part of his argument was disputed *at that time* by William

Thomson. He recognizes Rankine's peculiar genius and the physical *flair* that led him (for example) to insist on the distinction between resistance to change of bulk and to change of form; but plainly felt that he lacked the mathematical equipment demanded by the complexity of his notions. ("Luckily", he says, "Saint-Venant in his memoir of 1863 has given short and direct proofs of most of Rankine's results bringing out . . . their physical bearing.") William Thomson—in Rankine's life-time a close colleague—expressed a like opinion twelve years after his death. In the 20th of his 'Baltimore Lectures' (1884) Thomson said:—

I do not think I would like to suggest that Rankine's molecular hypothesis is of very great importance. The title is of more importance than anything else in the work. Rankine was that kind of genius that the names were of enormous suggestiveness; but we cannot say that always of the substance. We cannot find a foundation for a great deal of his mathematical writings, and there is no explanation of his kind of matter. I never satisfy myself until I can make a mechanical model of a thing.¹⁵

(Nor, I believe, did Rankine—though *his* models have not stood the test of time. For in his day 'model-making' was in fashion: having accepted atomic theory, a man would try to visualize a structure for the atom; if his work required it to have gyrostatic properties, he would picture it as a gyroscope—and both Rankine and Thomson did. Eddington has remarked on this tendency with a touch of condescension I confess that I could never understand. Is it really a great advance in physical thinking to set out equations without attempting to picture what they signify? to define waves as merely "the nominative to the verb 'to undulate'"?)

8. Thus in the one field of which I can speak with any assurance I find no really outstanding contribution which still has influence on our thinking. I wish that I could see the matter differently, for in Thermodynamics and in Fluid Mechanics I can make no claim to authoritative judgement, and there is risk in basing a judgement on the views of others—more especially when these others have written recently.

Because any attempt to assess what Rankine achieved must have regard to the times in which he lived; judgments may not be made in the light of things we take for granted, living now, if these things had still to be accepted when he wrote. Suppose that this were the year of his appointment, instead of its centenary: then Saint-Venant would have read his great *Mémoire*¹¹ before the French Académie in Coronation year, when our minds were otherwise preoccupied. For four more years Charles Darwin would still be working at his *Origin of Species*, leaving most of us still satisfied with the date given by Bishop Ussher for the world's Creation. Joule's paper on the Mechanical Equivalence of Heat¹⁶ he would have read to the British Association (had it been able to meet) in 1945; and by then little more than 25 years would have elapsed since the appearance of Sadi Carnot's *Réflexions*.¹⁷ That great event had occurred in Rankine's childhood; it would have come five years after the eclipse which first afforded a test of Einstein's theory. Dare we say that Carnot's notions were less revolutionary?—or absorbed less rapidly into the corpus of accepted theory, when we think of the work which William Thomson based on them and on the work of Mayer and of Joule?

I cannot feel altogether sure that Rankine appreciated the full importance of what had been achieved by Carnot. In the historical sketch which prefaces his textbook on Heat Engines¹⁸ he describes him as "son of the great Carnot", and the name occurs seldom in his chapter on the Principles of Thermodynamics, where these are presented as almost entirely due to Joule. (It was left for later writers—in particular, J. A. Ewing and H. L. Callendar—to present the work of Carnot in clear perspec-

ve.) Quite certainly, Rankine was not inclined to minimize any debt owed to an earlier investigator: his scrupulous recognition of Clausius' work is evidence of that. But I would guess that his reading, for the most part, concentrated on British work in his fields of interest; and that this was a not uncommon tendency in his time.

9. I shall say little of his work in Thermodynamics, for my knowledge of that subject goes no further than what Ewing's books gave me when I read for my Tripos of 1910, and I have studied his *Manual of the Steam Engine and other Prime Movers* mainly to gain an impression of his power of presentation. To be frank, I cannot rate it highly—and have thanked my stars that Ewing's books were written before I went up to Cambridge. But Rankine's book lacks clarity, I think, not only because the material was new but also because his interests had no bounds; not only because at the time no one could foresee in what directions theory would develop, but because he found *all* directions interesting—no one ever went more blithely 'down the garden path'. Is he concerned to instil the doctrine that heat is a form of energy? Yes, in Chapter III of his Part III (for already 40 pages have been given to General Mechanics, 258 to dynamometers and brakes, the strength of boilers and machine parts, the muscular power of men and oxen, of horses and mules, of hydraulics and water-power, to windmills, to measurement of temperature, to combustion and to fuels). There (§240), having said that heat is not directly measurable, he makes an assertion hardly likely (I would think) to ease the task of an ordinary reader:—"When a homogeneous substance is uniformly hot, every particle of it is equally hot; and every particle is hot in virtue of a condition of its own, and independently of forces exerted between it and other particles. *These are facts known by experience* [*italics mine*], and they lead to the following consequence:—That when the actual heat of a homogeneous and uniformly hot substance is considered as a quantity made up of any number of equal parts, all those equal parts are similarly circumstanced; and hence follows [*my italics, again*] *The Second Law of Thermodynamics: . . . the effects of those parts in causing work to be performed are equal.*" Having stated the Law, he expresses it in symbols; introduces the notions of absolute temperature and specific heat (real and apparent); then makes a statement which must have left many readers puzzled as to whether they are to regard its truth as *demonstrated*:—"The amount of actual heat, expressed in units of work, which corresponds, in a given substance, to *one degree of absolute temperature*, is the *real dynamical specific heat* of that substance, and is a constant quantity for all temperatures." How much simpler Clausius' statement of the Second Law—the impossibility, without an action of some external agency, of transferring heat from one body to another that is hotter? Ewing, though he says that the *Manual's* publication "formed an epoch",¹⁹ also observes that Rankine stated the Second Law in a form "neither easy to understand, nor obvious, as an experimental result, when understood".²⁰ I see it as the writing of a man tired by working overmuch.

10. Suspecting this to be a judgement based on ignorance, I went on to study his text-book *Applied Mechanics*,²¹ where, I thought, I would 'find the going easier'. But here too I am left wondering—despite my admiration of his industry—that in a book intended for use by engineers, and termed in its preface "an elementary manual", Rankine allowed his bent for "methodical classification" to make so hard the task of readers seeking help in particular problems. The nuggets are there; but how heavy the labour of mining them! The book may represent Rankine's notion of "a compact form"; but a book that runs to more than 650 pages is not mine!

I found in it one thing I was seeking; namely, an "abridgement" of his paper 'On the Stability of Loose Earth',²² (which Miller's 'Selection' does not include). I knew

(and have told tonight) what Saint-Venant and Timoshenko said about it; but now my aim was to learn what Rankine meant by abridgement. Hardly what we now take the term to mean, I think, for the shortening does not seem to me appreciable, though some "Lemmata" put in to make the paper self-contained were transferred to an earlier chapter of the manual (as is right, since their matter—the transformation of stress—has no special relation to earth-pressures). I have not counted words, but I estimate that the later presentation is, if anything, longer.

11. In both presentations Rankine claims to investigate frictional stability "without the aid of any artifice or assumption . . . such as Coulomb's Wedge of Least Resistance":²³ yet he adduces "Moseley's Principle of the Least Resistance"²⁴ which he regards as "a general principle of statics"; and when (in his paper) he attempts a general solution, he confronts the difficulty that it is not possible to distinguish (as the Principle requires) between "active" and "passive" forces—cause and effect. K. Terzaghi, reviewing Rankine's and Coulomb's work in the light of modern tests, concludes that neither theory is acceptable, but credits Rankine's with a "broader field of application".²⁵ Timoshenko²⁶ seems to me to have revealed the reason of their failure: namely that both, by omitting to consider strains in their granular material, left the problem indeterminate and accordingly insoluble—as problems in Strength of Materials were insoluble before Hooke's enunciation of his Law.

I have found nothing which suggests that Thomson ever took interest in this work, but Karl Pearson discusses it both under 'Rankine' and 'Saint-Venant'—I suppose, for the reason that subsequent investigators (and in particular Boussinesq, "the most distinguished pupil of Saint-Venant") extended in ways that took account of strains. Saint-Venant, he tells us, recognized Rankine's priority when M. Lévy, 10 years later (1867), rediscovered some of his results.²⁷ Boussinesq's work (which was published in the same year as Lévy's—in 1873,²⁸ when Rankine was dead) Saint-Venant, having made comparison with experiment, pronounced to be an improvement on all that had gone before.²⁹

Pearson's own account of the Royal Society paper³⁰ is restricted to those parts of it (the "Lemmata") which concern transformation of stress, and hence his *History*. He (like Timoshenko), derives the requisite formulae quite easily and briefly—calling them "elementary" as they were when he (*but not when Rankine*) wrote. He again refers to Rankine in his chapter on Boussinesq:³¹ giving Lévy credit (I am told, really due to Coulomb) for being the first to realize the importance of the conditions in the vicinity of a wall; citing a comment by Boussinesq on the hypothesis which (despite his preamble) Rankine used to solve his problem; and describing experiments made by Sir Benjamin Baker (and reported in the Inst. C.E.'s *Proceedings* of 1881) which indicated defects in Rankine's theory. Sir George Darwin also made experiments (reported in 1883), and these Boussinesq discussed—not wholly to Karl Pearson's satisfaction. (He ends his long account of Boussinesq's achievements with the sentences:—" . . . he has illustrated our subject by ingenious analysis rather than by special solutions" and . . . "Like the majority of the leading French mathematicians, Boussinesq is as a rule lucid in his analysis, if occasionally wanting in physical touch.")

Gilbert Cook, in a paper published some three months after his untimely death,³² remarks on Rankine's apparent lack of acquaintance with earlier work, and conjectures that he took but a fleeting interest in Soil Mechanics—that this, his only contribution, "was written to satisfy an intellectual impulse". Cook, as I do, finds the exposition lacking in clarity, and has little doubt of Rankine's having failed to

realize (as Coulomb and Lévy did) that the conditions obtaining in the vicinity of a wall are not (necessarily) those prevailing in the general mass. "He was fully cognisant of the property of cohesion, and its importance in a material such as clay, but the stability due to it he regarded as temporary only, liable to destruction by atmospheric action; and it would seem to be his view therefore that although his investigations concerned an ideal material, it was one to which actually earth masses would tend to approach, and such error as resulted from the use of his formulæ would be on the safe side."

12. The view to which I am led by my consultation of these and of other authorities is that the merit of this, Rankine's one contribution to Soil Mechanics, lies in his having recognized a problem (I have mentioned Saint-Venant's admission of his priority), not in his having left that problem solved. What he did was to investigate the consequences of an assumed criterion of stability for granular material such as sand, analogous with a 'yield criterion' for metals which distort (as we now say) plastically; namely, that the resultant stress on any plane must not have an inclination to the normal greater than the material's 'angle of repose'. The investigation entailed an analysis of stress which (in two dimensions) is simple by modern standards—known to every engineering student—and it might have been presented more concisely. But a century has elapsed since Rankine's work was done: it appeared only 30 years after Navier's general equations of elasticity.³³ Any way, this part of his work (his "Lemmata") was done correctly and has stood unchallenged since.

Almost certainly (and for a like reason) he overlooked another thing that we all realize today: that no criterion of the kind he postulated (that is, no one relation imposed on stress) by itself suffices for the solution of real problems—that account must be taken of displacements (strains). Had he recognized this, he might have recognized the likelihood of special conditions near a wall: having failed to do so, he was obliged to make the assumption (tacitly) that on every plane the resistance to sliding attains *at all points simultaneously* the limiting value which his theory contemplates. (Coulomb, more than 80 years earlier (1773),²³ had made a like simplification, but on better grounds, I think.) Judged by analogy with what is known now about stress-concentrations, it is an assumption which may be dangerous. But again we have to remember that we have no right to judge work in the light of later knowledge. How could Rankine, for example, have been aware of the importance of interstitial (seepage) water? Let Terzaghi (to whom, mainly, Soil Mechanics owes this knowledge) sum up the matter for us:—"... the fundamental misconception . . . does not reside in the theories [Rankine's and Coulomb's] as such. It lies in the failure of the designers to consider the limitations on the validity of the theoretical results."³⁴

13. Of what I suppose to be Rankine's greatest paper—"On the Thermodynamic Theory of Waves of Finite Longitudinal Disturbance"³⁵—the manual contains no like 'abridgment' (as was inevitable, since the paper came more than 10 years later), but some indications that his mind had turned in that direction as early as 1858 (the date of its preface).³⁶ It may be studied in *Miscellaneous Scientific Papers*, being—unlike the one relating to Earth Pressures—among the 37 papers in Millar's "Selection".

It is natural, I suppose, to be impressed most forcibly by work that by no stretch of imagination one can conceive of doing oneself? Anyway, if Rankine's fame had to depend on one only of his 153 papers, this, as it seems to me, is the one that it should rest on. For here is a problem both recognized and solved: to determine in what circumstances a wave of compression or of rarefaction is transmitted without

change of type (if you like—what manner of sound will suffer no distortion when its intensity compares with the ‘blasts’ of 15 years ago?). Keep the voice down to a mannerly noise-volume, and any sound will get through a speaking-tube undistorted: that must have been known long before Rankine’s day. But what happens if we treat the air more roughly—when a piston initially at rest is moved *suddenly* into quiet gas contained within a straight and smooth-walled tube? That was the problem to which Rankine addressed himself—at the outset contemplating “any substance, whether gaseous, liquid or solid, . . . the velocity of displacement of the particles being so great that it is not to be neglected in comparison with the velocity of propagation”; and he saw that the substance cannot assume a continuous velocity-distribution—that no adiabatic process induced by pressure only can entail a continuous change from one to another constant state.

I have found no more lucid assessment of his paper than that which Lord Rayleigh published in 1910,³⁷ and in which, arguing that the entropy of a closed system cannot decrease, he showed ‘rarefaction shocks’ to be impossible in a perfect gas. (It is noteworthy that the same volume of *Roy. Soc. Proceedings* contains this and another paper that reached like conclusions independently—the second of Sir Geoffrey Taylor’s papers, and the one with which he won the Smith Prize of 1910.³⁸) Rayleigh calls Rankine’s “a very remarkable investigation . . . which . . . has been much neglected”; and adds, in a footnote, “I must take my share of the blame”. He supposes Rankine, when he composed his paper, to have been unacquainted with the writings of his predecessors; but adds “. . . in a supplement he notices the work of Poisson, Stokes, Airy and Earnshaw” (not that of Riemann). Hugoniot, he says, “though he covers to a great extent the same ground, makes no reference to Stokes, Earnshaw, Riemann or Rankine”. His own review (as always) shows exhaustive reading and makes scrupulous attributions.³⁹

I have been greatly helped by his very clear account, for Rankine’s paper is by no means easy reading. It starts by introducing a concept—mass-velocity—which in a supplement he asserts to be “new, so far as I know” and “to have great advantages in point of simplicity”: namely, m , the mass of matter affected in unit time, which must have the same value everywhere, in a disturbance propagated without change of type. Rankine suggests *somatic velocity* as an alternative name for it, and expresses it in terms not of density but of its reciprocal—which he terms *bulkiness* and denotes by S . Rayleigh (who uses v in place of Rankine’s S) effects a simplification by bringing the wave to rest; as also does Taylor in his contemporaneous study³⁸—a remarkable note which, in the compass of 6 pages, answers Stokes’ query (regarding the possibility of discontinuities) in the same sense as Rayleigh, and derives an estimate of the thickness of shock waves in air.

Next, by a somewhat lengthy process, Rankine derives an equation that Earnshaw had obtained, expressing the constancy of his $(p + m^2S)$; and remarks that “no substance yet known satisfies it between finite limits in a disturbance occurring either isothermally or adiabatically, consequently during the disturbance there must be both change of temperature and heat-conduction”. (Here, Rayleigh remarks: “even under Boyle’s Law *viscosity* is competent to endow a wave with permanency.”) How, then, can the equation be satisfied? How much heat must be transferred? This question Rankine resolved in general terms, for *any* substance before (§9) he completed his solution in a special case—the perfect gas which had been so much discussed in recent years (he himself, in 1850, had related its specific heats)—and obtained (§11) the necessary relation, which shows that waves of finite condensation are propagated faster, waves of finite rarefaction more slowly, than an infinitesimal

disturbance. There is a definite lower limit to the velocity in a rarefaction wave, but no upper limit to the velocity in a wave of condensation.

But now comes a slip to which Rayleigh drew attention. The conditions of permanency require a definite transfer of heat by conduction, and heat cannot be thus transferred except from a hotter body to a colder; consequently no possible type of motion is reversible, and this consideration should have led Rankine to exclude periodic waves, also (as Rayleigh finds in a detailed study) waves of rarefaction—as entailing a transfer which would violate the second law of thermodynamics. But here again we have to remember when Rankine wrote. The objection can hardly have been realized by Thomson either, or he would have seen it as disposing of rarefaction waves more summarily than the fact (which Rankine tells us he had pointed out) of their being mathematically unstable. Taylor observes that neither Stokes nor Hugoniot nor C. V. Burton seem to have noticed that their equations relate to a motion which is irreversible.

14. Of Rankine's papers on Waves, Stream-Lines, Propellers, and the Lines of Ships, five are included in the *Miscellaneous Papers*; but of these I have not time to speak tonight, or of his Report on the Design and Construction of Masonry Dams—a severely 'practical' discussion, now in large part outdated.⁴⁰ This last appeared in the year of his death—too late for inclusion in *Applied Mechanics*. (It will call for notice at a discussion planned to take place next month in London,⁴¹ of the Dokan Dam now under construction in Iraq.)

Indeed, I must make an end of this survey, for my time is all but done. Of its utter inadequacy no one is better aware than I: yet how could so great a mass of books and papers, so many pages of closely-reasoned argument, be summarized within one hour? That I have 'spot-lighted' two or three items of Rankine's enormous output I justify on the ground that a complete review would entail not one lecture but a course! Equally I have had to focus attention on one or two facets of a many-sided personality, and I make no apology for having chosen those which seem brightest to my eyes. Of his writings, some still have value, others in course of time have been superseded: no man when he writes can foresee the course of Science. Of his influence on engineering there can be no doubt whatever; and believing that a man's influence derives from the aims which he sets himself, I went back yet again to his *Applied Mechanics*.

For the substance of its preface,⁴² as I said earlier, is the dissertation which, in obedience to your Senate, Rankine delivered at his installation; but translated from the Latin in which he spoke it, so within my comprehension. Its theme is "the evil influence (. . . the ghost of a defunct fallacy of the ancient Greeks and of the mediaeval schoolmen) of the supposed inconsistency of theory and practice upon speculative science", which "although much less conspicuous than it was in the ancient and middle ages, is still occasionally to be traced". "This it is which . . . leads scientific men" (he says) "sometimes . . . to state the results of really important investigations on practical subjects in a form too abstruse for ordinary use." ("This, but not this only", one of his readers would interpose, having wrestled with a number of his presentations.) But (he continues) its influence is "most fatal" on the practice of mechanics and engineering. Too often the strength and stability of a structure, which should result from skilful arrangement of its parts, are "supplied by means of clumsy massiveness, and of lavish expenditure of material, labour and money". Finally he writes of the unscientific inventor:—" . . . perpetual motion, or to speak more accurately, the inexhaustible source of power, is . . . the subject of several patents in each year."

A surprisingly doleful picture of the British engineering in his time, when its fame, we had thought, was world-wide! But later the picture brightens:—"... at the present time those evils show a decided tendency to decline . . . and that . . . harmony of theoretical and practical knowledge . . . which in former times was confined to a few remarkable individuals, now tends to become more generally diffused." He goes on to list Chairs instituted "with a view to promote the diffusion of that kind of skill . . . in the two Colleges of the University of London, in the University of Dublin, in the three Queen's Colleges of Belfast, Cork, and Galway, and in this University of Glasgow". Here he draws a parallel with Medicine which I have sometimes remarked on; also with the British Association (less than 25 years old, on the day we celebrate):—"What Section G is to Section A in the British Association, this class of Engineering and Mechanics is to those of Physics and Mathematics in the University."

15. Finally, "it being admitted that Theoretical and Practical Mechanics are in harmony", he goes on to consider in what ways their differences should be reflected in their modes of instruction. As always, he begins with classification:—"... three kinds: purely scientific knowledge,—purely practical knowledge—and that intermediate knowledge which . . . arises from understanding the harmony of theory and practice." He sets out the aim of each—at more length, perhaps, than we now would deem appropriate; giving this as the aim of the third (which I term Engineering Science):—"It qualifies the student to . . . adapt his designs to situations to which no existing example affords a parallel"; and emphasizing that this requirement calls for skills of a different kind from what pure science demands:—"In theoretical science, the question is—*What are we to think?* . . . But in practical science the question is—*What are we to do?* . . . we cannot allow our machines and our works of improvement to wait for the advancement of science; and if existing data are insufficient to give an exact solution . . ., that approximate solution must be acted upon which the best data attainable show to be the most probable." It is the last point he makes before a peroration very much to the taste, I would think, of a Victorian audience though perhaps a trifle flamboyant, judged by the standards of today. He ends with an eloquent tribute to James Watt.

16. I have quoted at this length from the dissertation, because it states so clearly what Rankine made his aims when he moved from "purely practical knowledge", his second category, to his third—which I would term engineering science. Could our own aims be stated better?—or the qualities that we should try to give our teaching? "... As far as is possible, mathematical intricacy ought to be avoided . . . simplicity is of the first importance. . . . An algebraical formula should only be employed when its shortness and simplicity are such as to render it a clearer expression of a proposition or rule than common language would be, and when there is no difficulty in keeping the thing represented by each symbol constantly before the mind."

Did Rankine achieve these aims? In respect of clarity, I cannot think so, unless—as I have said it ought to do—our assessment is based on the standards of his time. I do believe that in this regard the teaching of engineering science has improved in the intervening century; that students today, though they have more ground to cover, face a lighter task than they would have had to do had teachers not faced squarely their task of presentation. I have paid incidental tribute to J. A. Ewing's writings—a pleasure to read, whether technical or 'popular'; and I would also pay my tribute to C. E. Inglis, than whom I have never heard a better lecturer in his chosen field. Engineers are also in debt, I think, to men like Lamb and Love who,

though they wrote for mathematicians, would seem to have kept the needs of engineers in mind; and, in my judgement, most of all to Rayleigh, whose *Collected Papers*, bought many years ago, I treasure as an ultimate standard of perfection. His fifty working years (1869–1919) seem to me to have been the time in which—along with much else—clear scientific presentation was achieved. One is tempted to look for explanations.

17. I will hazard one tonight. Since Rankine's time, I think, men have come to see that this third kind of engineering is a 'border-line' subject, and that help in it can be found outside its frontiers. I myself for some 45 years, now, have watched this tendency. (For example, men concerned with stability of structures are all familiar now with 'Rayleigh's principle', and they have sought their knowledge at the fountain-head (his *Theory of Sound*) on the advice of their engineering teachers. No teacher advised me to do so, nor any of my books, when in 1910, I chose Elastic Stability as the topic of a fellowship dissertation.)

I do not suppose that Rankine advised his students to read the original papers of William Thomson—or that he would have been wise to do so, for Thomson too often find lacking in clarity (and I think I have read in Sylvanus Thompson's *Life* that his lectures could also be 'hard going')? Stokes too, though not obscure, seems to me over-deliberate—not to say verbose; and Clerk Maxwell, I understand, and Oliver Heaviside set their work out in ways that delayed full appreciation of it. I suspect that 'leisureliness' was a characteristic of their time, not of Rankine in particular—though harder things have been said of his writings than of theirs. Kelvin's considered judgement I have quoted: Karl Pearson supports it, writing (of a memoir published in the year of Rankine's death): "... like nearly all Rankine's papers, extremely suggestive, and rich in terminology, amounting in this case to very unnecessary verbosity".⁴³ Of his books, Timoshenko (dealing with his *Applied Mechanics*) writes:—"In his work Rankine prefers to treat each problem first in its most general form and only later does he consider various particular cases which may be of some practical interest. Rankine's adoption of this method of writing makes his books difficult to read, and they demand considerable concentration of the reader."⁴⁴

Yet he adds:—"Even now they are not completely superseded." And Karl Pearson summarizes:—"Such works as these of Rankine's and Weisbach separate very distinctly the first decade of our half-century from the previous thirty years. The step to them from books of the type of Tredgold's is very great and marks the beginning of the era of 'technical education'."⁴⁵ Finally, here is a modern assessment of Rankine by an engineer most qualified to judge (dear Gilbert Cook):—"He is acknowledged as one of the great pioneers in the movement to bring the powerful resources of mathematics and physical science to the practical problems of the engineer. It is to his efforts chiefly that we owe the success achieved in the struggle, in the nineteenth century, to give engineering an honourable place among the studies of our universities. Rankine embodied in himself that extremely rare if not unique combination of a professional engineer of wide experience, a mathematician and physicist of the greatest distinction, and an exponent of unusual power of the ideas and principles which constitute the framework round which engineering science is built."⁴⁶

18. Each and all of the views I have cited accord with what I have felt in the course of my own reading. What I could not feel to be right, on an occasion such as this, was to give and to quote only laudatory opinions. Again I say, to make a true assessment we must study the *ethos* of the times in which Rankine worked. Only so

shall we come to a full appreciation of what he did in transforming a practical craft to what engineering is today—an admixture of art and science.

In some ways he was fortunate; for even he and William Thomson, were they still active, could not now stake the claims they staked a century ago. The seams have been worked too thoroughly, more spade-work has to be done before a man can hope to find something new; though the tools now are more powerful, so that reading Rankine's papers one is struck, alternately, by his flashes of intuition and by the cumbrousness of the arguments that express them. One feels as when in face of some piece of Cyclopean masonry. How rudimentary the means! But, with our so vastly more powerful engines, what have we built that will bear comparison?

REFERENCES

General

- (A) Art. 'Rankine, William John Macquorn' (by G. C. Boase) in *Dictionary of National Biography*, vol. 47, London, 1896, Smith Elder and Co., pp. 290-2.
- (B) *Royal Society Catalogue of Scientific Papers* (19 vols. London: vols. I-VII (1867-77), H.M.S.O. Vol. VIII (1879), John Murray. Vols. IX-XIX (1891-1925), Camb. U. Press).
- (C) *Miscellaneous Scientific Papers by W. J. Macquorn Rankine* (edited by W. J. Millar, with a prefatory Memoir by P. G. Tait). London, 1881, Chas. Griffin and Co.
- (D) *Macquorn Rankine*: an oration by Sir James B. Henderson delivered in the University of Glasgow at the Commemoration of Benefactors, 15 June, 1932 (Glasgow, 1932, Jackson, Wylie and Co.). It cites "an article by Maxwell published in *Nature* in 1878". (Cf. note 7 to §4.)
- (E) 'Engineering': a lecture by Professor J. Small in a series delivered 1950-51 in the University of Glasgow, to commemorate the Fifth Centenary of its Foundation. (Reprinted from *Fortuna Domus*, 335-55.)

Introduction

- §1 1. 154 papers are listed in the Royal Society's *Catalogue of Scientific papers* (Ref. B), vols. v, vi and viii; but with one error (corrected in vol. viii). 127 papers appeared in the period 1853-73.
- 2. the preface to Ref. (C)—a biographical memoir. It was reprinted "with little amplification" from the *Glasgow Herald* of 28 Dec., 1872 (G. Cook; cf. note 32 to §11). Ref (A) cites two issues of the newspaper.
- §3 3. Cf. references, §8. Tait² gives Rankine's date as "late in 1849", Clausius' as "early in 1850". W. Wilson (Chap. I of *A Century of Science*: Hutchinson, 1951) gives Clapeyron credit for methods which "guided both Clausius and Thomson to the correct view of Carnot's principle . . . (about 1851)". In that year (Paper XVII of Ref. C) Rankine acknowledged indebtedness to Thomson for an explanation of Clausius' interpretation of Carnot.
- 4. Cf. references 22 and 28 (§§ 10 and 11).
- 5. Cf. references 35 and 39 (§13).
- 6. Rankine, W. J. M. *Miscellaneous Scientific Papers* (C), Nos. XXIX-XXXIII.

Personal Qualities

- §4 7. Cf. note 2 to §1. Ref. (A) cites other obituaries: in *Proc. R.S.*, vol. 21 (1873), pp. 1-4; *Proc. R.S.E.*, vol. 8 (1875), pp. 296-306; *Nature*, vol. 17 (1878), pp. 257-8.
- 8. Boswell, J., 1791. *The Life of Samuel Johnson, LL.D.* (Oxford U. Press: ed. 1933), vol. 2, p. 230 (ann. 1778).
- 9. "words and music alike being generally of his own composition." Tait's reference is to *Songs and Fables* (Glasgow: 2nd ed., 1874. Maclehose and Co.).

Contributions to Strength of Materials and to the Theory of Elasticity

- \$6 10. Cf. note 1 to §1. The paper was published in *Proc. Inst. C.E.*, vol. 2 (1843), 105-07.
- \$7 11. de Saint-Venant, B., 1853. *Comptes rendus*, vol. 36, p. 1028. The memoir appeared in *Mém. Sav. Étrangers*, vol. 14 (1856), pp. 233-560.
- On Rankine's contributions to the theory of frames and arches, cf. also Fleeming Jenkin's paper read (13 Dec., 1869) before the Royal Scottish Society of Arts.
12. Nos. III-VI in Ref. (C).
13. Cf. §§417-470 (Chap. XI, vol. 2) of Todhunter, I., and Pearson, K., 1893: *A History of the Theory of Elasticity* (Camb. U. Press).
14. *Proc. Phil. Soc. Glasgow*, vol. 6, No. 3 (Paper No. XIII in Ref. C).
- Other terms of Rankine's invention were less happy: 'tasinomic', 'thlipsinomic', and 'euthytatic' are three of many that have not survived.
15. Quoted by K. Pearson in *op. cit.* note 13, §1781 (vol. 3).

Contributions to Thermodynamics

- \$8 16. Joule, J. P. *Brit. Assoc. Rep.*, 1845, p. 31. *Phil. Mag.*, 27 (1845), 205-7 and 31 (1847), 173-76.
17. Carnot, S., 1824. *Réflexions sur la Puissance Motrice du Feu*, etc. (Paris: Bachelier. Republished, with a biographical notice, in 1878). Thomson (who first learned of Carnot's theory through a paper by Clapeyron, 1834) received a copy of the first edition from Professor Lewis Gordon, and based on this (1849) his 'Account' cited in note 18.
- The development of the principle of conservation of energy, and the birth of Thermodynamics as a science based on Carnot's theory (both within the decade 1845-55) are admirably reviewed in Chapter V of W. C. D. Dampier-Whetham's *History of Science* (1929, Camb. U. Press).
18. Rankine, W. J. M., 1859. *A Manual of the Steam Engine and other Prime Movers* (London and Glasgow. R. Griffin and Co.).
- In his paper of 1851 (No. XVII of Ref. C) Rankine credits Clausius and W. Thomson with "the merit of combining Carnot's Law, as it is termed, with that of the convertibility of heat and power"; but adds, "I have not yet seen his [Thomson's] paper, nor become acquainted with the method by which he proves Carnot's Law". What I find perplexing is that 8 years of further study (in proximity to Thomson) should have resulted in no more, in the 'historical sketch', than 10 lines based on Thomson's 'Account of Carnot's Theory' (1849).
- \$9 19. Ewing, J. A. *The Steam-Engine and other Heat-Engines* (3rd ed., 1910, Camb. U. Press), §23.
20. *Ibid.* §78. Tait² says of Rankine's statement:—"Its actual meaning is inscrutable."

Contributions to Soil Mechanics

- \$10 21. Rankine, W. J. M., 1858. *A Manual of Applied Mechanics* (London. Chas. Griffin and Co.).
22. Rankine, W. J. M., 1857. *Phil. Trans. R.S.* (A) 147, 9-28.
- \$11 23. Coulomb, C. A., 1773. *Mém. par divers Savants*, 7, 343-82.
24. Moseley, H., 1833. *Phil. Mag.*, vol. 3, pp. 285-8.
25. Terzaghi, K., 1936. *J. Boston Soc. Civ. Engineers*, 23, pp. 71-88. (Reprinted in *Contributions to Soil Mechanics*, Boston, 1940.)
26. Timoshenko, S. P., 1953. *History of Strength of Materials* (McGraw-Hill Co.), §68 (p. 332).
27. Levy, M., 1873. *J. de Liouville* (2 sér.), 18, pp. 241-300.
28. Boussinesq, J. 1873 *Comptes Rendus*, 77, pp. 667-71 and 1521-5. Cf. *op. cit.*, note 13, §1570-1626.
29. *Op. cit.* note 13, §242.
30. *Ibid.* §453.
31. Cf. note 28. The attribution to Levy is in §1597; the comment on Rankine's hypothesis in §1604, and on G. H. Darwin's experiments in §1609. B. Baker's experiments are noticed in §1606.
32. Cook, G., 1951. *Géotechnique*, vol. 2 (4), pp. 271-79.
- \$12 33. Navier, C. L. M. H., 1821. *Mém. de l'Institut*, vol. 7 (1827), pp. 375-93.
34. Terzaghi, K., *loc. cit.* note 25 (§11), p. 86.

Shock Waves (Rankine-Hugoniot Theory)

- §13 35. Rankine, W. J. M., 1870. *Phil. Trans. R.S. (A)* vol. 160, pp. 277-88 (Paper No. XXXII of Ref. C). His exposition was followed closely by J. H. Poynting in his article 'Sound' for the 11th edition of the *Encyclopaedia Britannica* (1911). (Camb. U. Press.)
36. e.g. in *Applied Mechanics*, §416-20 and 647.
37. Rayleigh, Lord, 1910. *Proc. R.S. (A)*, vol. 84, pp. 247-84; *Scientific Papers*, No. 346 (vol. V, pp. 573-610).
38. Taylor, G. I., 1910. *Proc. R.S. (A)*, vol. 84, pp. 371-77. This paper (apparently stimulated by an objection adduced by Rayleigh (*Theory of Sound*, vol. 2, p. 41) against a conclusion drawn by Stokes from Poisson's equation) makes no reference to Rankine's. H. Lamb (*Hydrodynamics*, §284) cites it and Lord Rayleigh's paper, also a presentation by J. Hadamard (1903) of Hugoniot's investigations.
39. Rayleigh's paper cites Poisson (1808), Stokes (1848), Airy (1849), Earnshaw (1859, 1860), Riemann (1860), Burton (1893), Hugoniot (1887), Weber (1901) and Duhem (1909), but is mainly concerned with Rankine's paper (cf. note 35), which (it says) was the first to take account of heat-conduction. "... Although there are one or two serious deficiencies, not to say errors, ... the memoir makes a very definite advance."
- §14 40. The Report is numbered XXXIV in Ref. C, where a footnote says it is "from *The Engineer* for Jan. 5th, 1872". Professor A. J. Sutton Pippard, in the centenary issue of that journal (1956), gives it high praise.
41. At an Ordinary Meeting of the *Inst. C.E.* on 17 Jan., 1956. (Cf. *Chartered Civil Engineer*, Nov., 1955, p. 26.)
42. A footnote states that it also includes the substance of Rankine's inaugural lecture, delivered on 3 Jan., 1856.
43. *Op. cit.* note 13, §455.
44. *Op. cit.* note 26, §44. (Timoshenko makes a like comment on the writings of Grashof.)
45. *Op. cit.* note 13, §470.
46. *Loc. cit.* note 32, p. 271.

Professor A. J. S. Pippard, in proposing a vote of thanks to Sir Richard Southwell, recalled that when he had been an undergraduate his old Professor, Robert Muir Ferrier, had reserved a special tone of voice and, he thought, a special attitude of mind when referring to Rankine. He had always given Rankine his full name—William John Macquorn Rankine—and always referred to him as "that great man". There was no doubt that his professor had had a veneration for Rankine which he had passed on to his students.

Professor Pippard said that he, for one, had always felt that Rankine was one of the great giants of his period, and of any period, in engineering science. A few years ago, just after the war, he had had a slight thrill when the Librarian at the Imperial College Engineering Library had come to him one day and said that, in turning over some of Unwin's books, he had unearthed a short note from Rankine to Unwin, stuck in the miscellaneous collected papers of Rankine. The note was not much in itself, but it showed that even the great could slip up on occasion. It read:—

"59, St. Vincent Street,
Glasgow
23 June, 1869

Dear Sir,

I beg you will accept my best thanks for the volume you have been so kind as to send me. I shall value it highly and read it with much interest.

I have also to thank you for having pointed out the oversight in Figures 59, 60 and 61 of my book on prime movers. The difference of dimensions, however,

was not so great as you suppose, for the scale of Figure 59 is $\frac{1}{200}$ and that of Figure 60 is $\frac{1}{330}$.

I am, dear Sir,

Yours,

W. Macquorn Rankine."

So even the great ones slipped up over scales!

Sir Richard Southwell had given them what, from his own knowledge of Rankine's work, he considered to be a very just estimate of that work and of the man. He thought that anyone who had tried to read Rankine would have no doubt about the justness of Sir Richard's judgement as to the obscurity. He had tried to read a few of Rankine's papers, and they were certainly that.

This estimate came from a man who had himself in no small manner contributed to engineering science today. At the beginning of the meeting they had heard of Sir Richard's contributions and his work—his first work on elastic stability, followed by a long series of papers on elasticity; and then, in more recent years, his completely new relaxation procedure put him into the front rank of engineering scientists. It had indeed been a very great privilege to hear the opinion of a man of that calibre upon one of his predecessors of 100 years ago.

He had very great pleasure in proposing a very hearty vote of thanks to the Rankine Memorial Lecturer.

The vote of thanks was accorded with acclamation, and the meeting terminated.

SPECIAL GENERAL MEETING

Monday, 16 January 1956

WILLIAM KELLY WALLACE, C.B.E.,
President, in the Chair

The notice convening the meeting was taken as read.

The President stated that the meeting had been requested by twenty-four Corporate Members under By-Law 103.

The following Resolution was moved by Mr W. S. Catlow, and seconded by Mr M. T. Shaw:

"This meeting of Corporate Members of The Institution of Civil Engineers is of the opinion that the current By-Laws and Regulations are out of date in that they allow the admission of Corporate Members who have not passed the Institution Examination or other examinations recognized by the Council as giving exemption thereto. Such is contrary to modern practice and tends to lessen the professional status of Chartered Civil Engineers when compared with that of Corporate Members of other Professional Institutions.

"In accordance with para. 117 of the By-laws and Regulations the Council are directed to take action to alter or revoke these By-laws and Regulations to this effect and in particular to revoke sub-para 3 (iii) (B) and (C)."

The following amendment to the Resolution was moved by Sir Arthur Whitaker, and seconded by Mr J. H. W. Turner:

"This meeting of Corporate Members of The Institution of Civil Engineers requests the Council to consider whether the existing By-laws governing election direct to full Membership are out of date and, if so, to make proposals to amend them."

The following took part in the discussion:

Mr N. E. M. Bryden
Mr H. Shirley Smith
Mr E. F. Hope-Jones
Dr H. Q. Golder
Mr D. N. Mitchell
Mr G. A. Phillips
Mr F. Newhouse
Mr H. F. Cronin
Mr A. C. Hartley

Mr H. R. Lupton
Mr B. J. Thorpe
Professor A. W. Skempton
Mr P. G. K. Knight
Dr T. P. O'Sullivan
Mr A. S. Quartermaine
Mr J. H. Jellett
Mr H. P. T. Lind
Sir Herbert Manzoni

A vote was taken upon the amendment by a show of hands, and the Chairman declared that the amendment had been carried by 183 votes to 41.

ORDINARY MEETING

17 January, 1956

WILLIAM KELLY WALLACE, C.B.E., President, in the Chair.

The Council reported that they had recently transferred to the class of

Members

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|---|--|
| DLINGTON, ALEC. | SMITH, JOHN GERALD. |
| OGDON, RONALD STEPHENSON. | TAYLOR, OLIVER MIDGLEY, M.A. (<i>Cantab.</i>). |
| MILTON, BASIL LONG, M.A., M.A.I. | WORTH, ROBERT EDWARD JAMES, B.Sc. |
| (<i>Dublin</i>). | (<i>Eng.</i>) (<i>London</i>). |
| MORRISON, MALCOLM HALLEY, B.Sc. (<i>Edinburgh</i>). | |

and had admitted as

Graduates

- | | |
|---|--|
| ADNESEN, LARS, B.Sc. (<i>Durham</i>), Stud. I.C.E. | BURROWS, EDWIN RAYMOND, B.Eng. (<i>Liverpool</i>), Stud.I.C.E. |
| ALLWOOD, ROGER JAMES, B.Sc. (<i>Birmingham</i>), Stud.I.C.E. | CAMPBELL, JOHN WILLIAM EDWIN, B.Sc. (<i>Eng.</i>) (<i>London</i>), Stud.I.C.E. |
| ASPIN, BRIAN HELLIWELL, B.Sc.(<i>Eng.</i>) (<i>London</i>), Stud.I.C.E. | CARTER, JOHN LISTER, Stud.I.C.E. |
| AYLIFFE, NORMAN RICHARD, B.Sc. (<i>Durham</i>). | CHUA NGOH CHUAN, Stud.I.C.E. |
| BACK, PAUL ADRIAN AUCHMUTY, B.Sc. (<i>Cape Town</i>). | COOKE, PATRICK JOSEPH, B.E. (<i>National</i>). |
| BARRELL, JOHN LEONARD, Stud.I.C.E. | DANBURY, MICHAEL FRASER, B.Sc. (<i>Birmingham</i>), Stud.I.C.E. |
| BARRETT, BRIAN ALBERT. | DEANS, KENNETH JOHN, B.Sc.(<i>Eng.</i>) (<i>London</i>), Stud.I.C.E. |
| BARRY, MICHAEL AUGUSTINE, B.E. (<i>National</i>). | DICK, DAVID, B.Sc. (<i>Edinburgh</i>), Stud. I.C.E. |
| BAZLEY, REGINALD CHARLES, B.Eng. (<i>Liverpool</i>), Stud.I.C.E. | DUTTA-MUNSI, TUSHAR KANTI, B.E. (<i>Calcutta</i>), Stud.I.C.E. |
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The following Paper was presented for discussion and, on the motion of the President, the thanks of the Institution were accorded to the Authors.

Paper No. 6113

THE EXPERIMENTAL AND MATHEMATICAL ANALYSIS OF ARCH DAMS, WITH SPECIAL REFERENCE TO DOKAN

by

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SYNOPSIS

The Paper describes an investigation of the design for the Dokan dam carried out for Messrs Binnie, Deacon & Gourley, Consulting Engineers to the Government of Iraq Development Board.

The stress analysis of an arch dam is a problem of considerable complexity and demands lengthy computation. The most elaborate treatment so far published is due to the U.S. Bureau of Reclamation and is known as the "trial load" method. Its successful application requires a staff of experienced computers, and since these were not available a completely different approach was made which resulted in a new combined analytical and experimental treatment of a very difficult problem.

The process known as "relaxation" enables mathematical equations to be solved numerically to any desired degree of accuracy, and although no three-dimensional problem comparable in difficulty with the arch dam had hitherto been attempted, a skilled team at the Imperial College was prepared to undertake the mathematical part of the study.

The exact elastic equations for the dam were formulated in terms of displacements instead of the more usual stress functions, and were solved by relaxation. To obtain an approximate answer within a reasonable time it was assumed at first that the dam was a section of a complete cylinder, with a wall of the same profile as the dam at its greatest depth. Symmetry of structure and loading then enabled certain terms in the equations to be neglected and the problem was reduced to one of two-dimensional stress. This calculation, which became known as the "tumbler" solution, was, however, insufficiently accurate on account of abutment effects, and a complete three-dimensional relaxation analysis was undertaken.

To obtain guidance upon the behaviour of the dam and also to check numerical results of computations, scale models of the dam were cast in rubber and systematically studied. The models were loaded by water, and the low value of Young's modulus for the rubber made it possible to measure with considerable accuracy displacements on both faces of the dam in three mutually perpendicular directions.

Calculations were made of the stresses due to gravity loads and water pressure acting separately and also of those due to variations in temperature.

The dam will be built as a number of separate monoliths and stresses due to gravity will not be affected by arch action, which can only operate after contraction joints are grouted. Relaxation calculation of these gravity stresses confirmed the results of an earlier investigation that the assumption of a linear distribution of stress on horizontal

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sections gives results very close to those obtained by exact analysis. Stresses due to water pressure were determined from the full three-dimensional analysis, and those due to variations of temperature by the tumbler analysis for the centre section of the dam, and by a more conventional *encastré* arch-rib analysis for the abutments.

The stresses throughout the dam for the various conditions of loading are exhibited in a series of diagrams, and a comparison is made between the results of the experimental and analytical determinations of displacements. The mathematical arguments relating to the relaxation treatment are given in the Appendix.

INTRODUCTION

THE Paper describes an investigation made in the Imperial College of Science and Technology, London, for Messrs Binnie, Deacon & Gourley, Consulting Engineers to the Development Board of the Government of Iraq, in connexion with the design of a large concrete arch dam which is now under construction at Dokan on the Lesser Zab River.

The work, carried out partly in the Department of Civil Engineering and partly in the Relaxation Division of the Department of Mathematics, was begun in January 1952 and took rather more than 3 years to complete.

Since it was essential that the contract drawings for the dam should be completed by a specific date, a number of problems which arose out of the work were necessarily

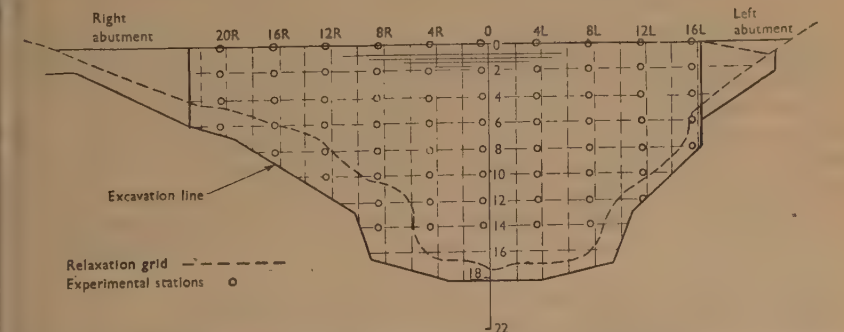


FIG. 2.—DEVELOPED SECTION ALONG VERTICAL WATER FACE, SHOWING GRID REFERENCE SYSTEMS

left for future study. It is not claimed, therefore, that this Paper provides answers to more than a few of the many questions presented by arch dams in general; it does, however, describe a new mathematical and experimental approach to such questions, and one which supplied the essential information for the design of Dokan dam. This dam is to be of concrete with a constant radius of 120 m at the water face. A developed elevation of this face is shown in Fig. 2 and the profile of greatest depth in Fig. 5. Fig. 1, facing p. 208, is from a photograph of a model of a preliminary design placed in the correctly contoured valley and gives a general impression of the scheme. It also serves to define certain dimensions and terms used in the Paper.

After excavation of the valley the lower half of the dam structure will be very nearly geometrically symmetrical about the vertical plane through O in Fig. 2. Model experiments, to be described later, emphasized this point and indicated clearly that the dam behaved elastically as a symmetrical structure.

The Dokan Valley consists of limestone and dolomite. The arch proper rests on a substantial concrete block; its upper part abuts against the concrete of gravity wing walls, and the lower part directly on the dolomite. Nowhere does the arch come into direct contact with the limestone.

THE THEORETICAL PROBLEM OF THE ARCH DAM

Suppose that a dam of circular-arc plan form and constant height simply rests in a valley and has no shear connexion with the base or abutments, which are for the moment considered to be rigid. Under the action of gravity forces and water pressure all vertical sections are similarly stressed and it is only necessary to determine the values of the stresses at every point in one such vertical section to obtain a complete solution; the stress system will consist of vertical stresses, complementary shearing stresses in the plane of the section, and tangential or hoop stresses. If, however, the base is assumed to be rigidly connected to the foundation, the problem is complicated by the imposition of this extra constraint which entails the development of radial shearing stresses between the base of the dam and the rigid foundation. All elementary vertical strips are now in the same condition as that assumed in the analysis of a gravity dam idealized for design purposes with, however, the addition of hoop stresses.

The assumptions made so far allow the end sections of both dams to slide relatively to the abutments, but, whilst the first dam can also slide on the foundation, the second cannot do so and will deflect as a cantilever about the fixed base. If, in addition to the base fixing, the dam is connected to the rigid abutment rock, all movements of the end sections will be prevented, the state of stress will again be modified, and elementary vertical strips will no longer all behave similarly.

The centre strip of the idealized dam thus connected to foundation and abutments will, from considerations of symmetry, be free from shearing forces on its vertical faces. The end sections will, however, be subjected to shears, and there will be a steady transition from the state of stress existing at the centre section to that at the abutments. A complete stress analysis now demands the determination of component stresses at all points in the structure and not simply those in one typical vertical section.

The practical problem is generally considerably more complicated than that of the idealized structure outlined, since few valleys are of constant depth and most are irregular and unsymmetrical. If a dam in such a valley is assumed to be divided into a number of elementary vertical strips each of which is analysed on the assumption that the vertical faces are free from shear stresses, the resulting strains will be incompatible, i.e., the strained vertical face of one section will not be exactly the same shape as the contiguous face of the adjacent section. In the actual structure, strain compatibility is attained by the development of shearing stresses between adjacent strips.

Another complication arises from the fact that the bed-rock is not rigid but has elastic properties which are not usually very accurately known. They may vary, perhaps considerably, from point to point, but even if they were known to be uniform, the strains in an irregular abutment mass due to loads transmitted through the dam could not be accurately calculated; yet upon these strains will depend the stresses, not only in the abutments and foundations but to a certain extent in the dam itself.

So far it has been tacitly assumed that the only factors contributing to the stresses in an arch dam are its weight and the water pressure, but the effects of temperature variations, uplift, and possibly earthquake shocks all need consideration.

Temperature stresses

The behaviour of a concrete dam arising from changes in its temperature cannot be foretold with any precision, owing (among other factors) to uncertainty about the physical properties of the concrete; estimates of stresses must, therefore, be of very doubtful accuracy even if the variations of air and water temperatures at the dam site are accurately known.

If the faces of a dam were exposed sufficiently long to constant but unequal temperatures, a stable state would be attained and the temperature distribution in the mass could be estimated by classical methods. Conditions do not, however, remain constant sufficiently long to ensure this but are subject to diurnal and seasonal variations. It is a common assumption that the temperature through the dam varies linearly; this is equivalent at any level to a uniform change through the whole thickness superimposed upon a temperature varying uniformly between equal positive and negative values on the two faces.

Usual conventional methods of calculation assume that the dam may be treated as a series of *encasté* arches each subjected to appropriate thermal changes. A uniform rise in temperature above the assumed normal tends, in general, to ameliorate the stresses caused by water pressure; a uniform drop tends to aggravate them. A uniform temperature gradient varying from a positive value on one face to an equal negative value on the other face produces no circumferential strain if the abutments are rigid, but the prevention of the natural alterations in fibre lengths which would occur under free conditions may give rise to considerable stresses.

Stresses calculated on the assumption of rigid valley walls are relieved in practice by elastic abutment movements and whilst the extent of this relief cannot be estimated accurately it is certainly appreciable.

Uncertainties arising from lack of exact knowledge of the coefficient of thermal expansion and Young's modulus are so great that if extreme high and low values for those physical properties are assumed, the calculated stresses due to a specified temperature variation may differ by as much as three to one. Further, concrete creeps under stress and while this relieves stress, it is impossible to allow for its effect accurately.

Although the assumed steady temperature state may never be reached, high stresses can be occasioned by local changes in temperature. When, for example, a portion of the structure is heated or cooled, surface stresses will be induced before the inner layers are affected, since the free straining of the exposed layer is prevented.

Stresses due to uplift

In the absence of adequate drainage a dam might well be subjected to uplift pressures varying in intensity from the static head at the water face to zero at the air face, but since at Dokan an efficient system of drainage is to be provided the effect of uplift has not demanded special consideration.

Stresses due to earthquake shock

Serious earthquakes are not anticipated in the Dokan region, but a dam in an active seismic area may be subject to horizontal accelerations of varying frequency and intensity applied in any direction, and at best the estimate of the effects of such disturbances is almost a matter of guesswork.

Conventional methods assume an up-and-down stream acceleration of $0.1g$ with a period of 1 sec and the effect of this on the mass of the dam and also on the

water in the reservoir can be approximately calculated.¹ Such calculations for Dokan indicate an increase of about 20% above the stresses due to water loading only. The effects of shock in a cross-stream direction are even more difficult to predict, but general considerations indicate that the Dokan structure is adequate for any earthquake likely to occur.

The foregoing considerations show only too clearly that excessive refinement in analytical treatment is unjustifiable and leads only to a fictitious accuracy which may even, if interpreted too rigidly, result in a false sense of security. For it must be emphasized from the beginning that no mathematical procedures can take account of lack of precision in data. Nevertheless, a complete solution of the stress equations under carefully specified conditions provides a good indication of the adequacy of the structure and also a valuable standard by which approximate methods of analysis can be judged. The main object of the present investigation was to obtain such a solution.

Numerous attempts have been made to formulate approximate methods of analysis of sufficient accuracy for design purposes but none of them has gained general acceptance. Jaeger² has described some and Bourgin³ has treated others in detail. Several attempts to find satisfactory engineering approximations have been made already in the course of the present investigation and some of these are promising. It is hoped at a later date, after further research, to present suggestions for preliminary design methods of a reasonably simple type.

THE TRIAL-LOAD METHOD

The most elaborate and detailed method of analysis so far developed is the trial-load method devised by the U.S. Bureau of Reclamation after many years of systematic research.⁴ This method visualizes the dam as consisting of two coexistent and interpenetrating structural systems; one, a series of vertical cantilevers, and the other, a series of arch rings. The water loading is divided by repeated trial between the two systems in such proportions that the displacements of both are equal at a number of selected points.

The method as now used by the engineers of the Bureau of Reclamation is based on the following assumptions:⁵

1. The rock formations which make up the foundation and abutments at the site are homogeneous and uniformly elastic in all directions, and are strong enough to carry the applied loads with stresses well below the elastic limit.

2. The concrete in the dam is homogeneous and uniformly elastic in all directions and is strong enough to carry the applied loads with stresses well below the elastic limit.

3. The dam is thoroughly keyed into the foundation and abutment rock throughout its contact with the valley profile so that the arches may be considered fixed in relation to the abutments, and the cantilevers fixed in relation to the foundation.

4. The vertical construction joints in the dam are grouted, or the open joints filled, before the water load is applied, so that the structure may be considered to act as a monolith and arch action to begin as soon as the reservoir begins to fill.

5. The total vertical loads, including the weight of the water on the faces of the dam as well as the weight of the concrete, are assigned to the cantilever

¹ The references are given on p. 230.

elements and are assumed to be transmitted to the foundations without any transfer of load laterally to adjacent cantilevers by means of vertical arching.

6. The effect of creep in concrete can be adequately allowed for by using somewhat smaller values of Young's modulus than would otherwise be adopted.

These general assumptions have been accepted in the present analysis and additional particular assumptions made in regard to Dokan:

7. Geophysical tests of the valley indicated that the dolomite had an average Young's modulus of 7,500,000 lb/sq. in. parallel to the bedding planes and 3,500,000 lb/sq. in. normal to these planes. The instantaneous value of the modulus for concrete is about 3,500,000 lb/sq. in. and it has therefore been assumed that the rock foundation for the dam and the abutments has the same modulus as the concrete of which it will be built. To allow for creep, etc., this has been assumed to be 2,000,000 lb/sq. in. (141,000 kg/sq. cm). Poisson's ratio has been taken to be 0.15 and the density of the concrete to be 150 lb/cu. ft. (2,400 kg/cu. m).

8. The temperature data for the evaluation of thermal stresses can be taken from measurements recorded at Boulder Dam owing to the similarity in climatic conditions at Dokan and Boulder.

RELAXATION METHOD OF ANALYSIS

Experience in application of the trial-load method was lacking in the United Kingdom and a new approach was decided upon, using relaxation analysis.^{6, 7} It was realized that this might well require as much computation as the trial-load method, but it had the advantage that experts in the technique were available.

On the basis of the assumptions stated above it is possible to formulate the exact equations governing the components of stress at any point in the arch dam. The relaxation method of solving these equations is numerical and employs certain mathematical approximations; the answers it achieves are therefore essentially approximate, but, given time, this approximation can be brought as close to an exact solution as may be desired. The artificial device of dividing the dam into interpenetrating cantilevers and arches and then determining the stresses from resultant actions by formulæ of doubtful validity then becomes unnecessary. It was, however, realized that a complete solution would take a long time and that approximate answers would be required quickly to provide urgently needed design data. The development of a simplified treatment was therefore the first objective but before describing this it is desirable to envisage the complete problem.

The complete three-dimensional problem

The dam is a three-dimensional body and requires the determination of a three-dimensional stress-system. Experience showed that the work was best carried out in terms of the three components of displacement at any point in the structure. A cylindrical-polar system of co-ordinates, r , z , and θ , as indicated in Fig. 6a, was adopted and the displacements of any point denoted by u (radial), v (vertical) and w (tangential), each taken positive in the corresponding positive direction of the co-ordinates r , z , and θ respectively. At any point there are six components of stress each defined by two letters, e.g. $\widehat{rz}$ acts in the direction of r on a plane normal to z ; it is a shear stress and is equal in magnitude to the complementary shear stress $\widehat{zr}$ which acts in the direction z on a plane normal to r . Direct or normal stresses

such as $\widehat{zz}$ (which is a stress acting in the direction of z on a plane normal to z) are shown in Fig. 6a and the three sets of complementary shear in Figs 6b, c, and d.

At any point the six components of stress can be expressed in terms of the three displacements by the relations (1) given in the Appendix. Thus, once the displacements have been determined, values of the stress components may be deduced directly from them.

The displacements themselves are found by solving the three governing differential equations (2) which result from substitution of the expressions (1) for the stresses into the equations of equilibrium for the point. These equations have to be solved, subject to stated boundary conditions, to determine values of u , v , and w throughout the dam and the adjoining rock foundation.

The relaxation net

The relaxation solution of equations (2) is numerical and provides actual values of the displacements u , v , and w at a number of discrete points throughout the dam and adjoining rock. These points are the nodes of a suitably chosen three-dimensional network or grid system. The three sets of mesh surfaces chosen were horizontal planes ($z = \text{constant}$), vertical planes ($\theta = \text{constant}$), and cylindrical surfaces ($r = \text{constant}$). The mesh lengths were 12.4 m in the vertical direction, 6.2 m in the radial direction and in the angular direction 12.4 m at the water face (radius = 120 m): correspondingly $12.4r/120$ m at any other radius r .

Even when, in the light of the experiments on the rubber model, the assumption had been made that there was a vertical plane of elastic symmetry in the dam so that it would only be necessary to find the displacements in one half, the net still contained some 350 nodal points, at each of which the solution had to determine the three components of displacement to a sufficient degree of accuracy to deduce stresses to within 10% of their accurate values. It will be appreciated that there was no virtue in carrying the work to a greater degree of accuracy in view of the uncertainty in the original data.

Simplified assumptions: "tumbler" solution

Dokan Dam is to be built in sixteen monoliths—five in the abutments and eleven in the actual arch. Contraction joints between these will not be grouted until the concrete temperatures have been lowered to specified values. Gravity effects are thus confined to the monolith concerned, and, therefore, since differences in gravity displacements between adjacent monoliths cannot produce shearing stresses on vertical cross-sections, it seemed reasonable in an approximate treatment to neglect $\widehat{z\theta}$ stresses.

Again, as seen in Fig. 2, the dam has a central portion of maximum constant height. Somewhere within this portion the resultant radial shearing force will be zero, and so it was decided to neglect also in the first instance the $\widehat{r\theta}$ stresses on the centre section and to make the analysis on the assumption that the only operative stresses were the radial, vertical, and hoop stresses $\widehat{rr}$, $\widehat{zz}$, and $\widehat{\theta\theta}$ respectively, and the complementary shear stresses $\widehat{rz}$ and $\widehat{zr}$. This assumption is reasonable for the centre section but the omission of $\widehat{r\theta}$ stresses is not justifiable for others; the argument for the neglect of $\widehat{z\theta}$ stresses is still valid.

The approximate method based on these assumptions was a plane two-dimensional relaxation solution; a vertical strip of the dam was considered to be a portion of a complete circular tube held at the base section and subjected to external water

pressure, the wall of the tube being the same shape as the dam profile at its greatest depth. This became known as the "tumbler" solution, but although it was useful for some purposes and, as it proved, gave a good estimate of the vertical stresses, it was far from complete and took considerable time to compute.

In view of the uncertainties attaching to the approximate results obtained by the tumbler solution, it was decided to undertake, also, a complete three-dimensional relaxation analysis, a decision of considerable interest since it was the first occasion on which the analysis of a mass structure of this size and complexity had been attempted other than on the basis of drastic simplifying assumptions.

At first, stress functions were used in the relaxation analysis, but it soon became apparent that it would be much better to recast the equations in terms of displacements, one important consideration being that results obtained from model experiments could be used directly in conjunction and for comparison with the analysis.

The necessary mathematical arguments are given in the Appendix and it need only be mentioned here that relaxation is a systematic trial-and-error method. The structure to be analysed is defined by the space grid or mesh and, as a start, values are assumed for the displacements of all node points on this network. These values will not be correct, but a sufficient number of systematic corrections or relaxations will lead to a determination of the correct values. The accuracy of the first guess will naturally influence the amount of calculation required to obtain the true result, and so any help in making the initial guess a better approximation to the final result is clearly of value. Experiments on rubber models provided estimates of the component displacements at a number of points on the surface of the dam and these served as guides and checks for relaxation and other analyses.

MODEL EXPERIMENTS

The physical properties upon which the behaviour of a dam depends are the density of the material of which it is built, the value of Young's modulus, and Poisson's ratio. If a model $1/n$ full size were made in the same material as the dam, the displacements of corresponding points on the prototype and model would be in the ratio $n^2:1$ and the stresses and strains in the ratio $n:1$. The model would necessarily be small compared with the full-size structure (i.e., n would be large) and its response to load would be insufficient for accurate measurement. There are certain obvious ways of overcoming this difficulty; the model could be made of material having a lower value of Young's modulus than the prototype, pressures could be applied by mercury instead of water, and either or both gravity and pressure forces could be reproduced mechanically in the correct proportions on a model of sufficient flexibility for accurate measurements to be made.

One criterion which should theoretically be satisfied is that Poisson's ratio for the material of both model and prototype should be the same. What is ideally needed, therefore, is a material which can be moulded or cast, has a low value of Young's modulus, a Poisson's ratio of 0.15 or thereabouts, and exhibits a linear load-displacement relation. Its density is of minor importance since the gravity and pressure displacements can be found independently and superposed as required. Rubber fulfils these requirements except that Poisson's ratio is 0.5 instead of 0.15; the effects of the difference could not easily be foretold but it was decided, after consideration of the various possibilities, to use this material for the experimental research.

The model dams were cast by Rubber Technical Developments Ltd of Welwyn,

using a recently developed technique. The results were excellent, the material having a good linear stress/strain relation and a Young's modulus ranging from 110 to 200 lb/sq. in. for different models.

The first two experimental pilot models P.M.1 and P.M.2 shown in Figs 7 and 8, facing p. 208, were simple idealized structures of trapezoidal section of the same profile as a preliminary design, D.1 (Fig. 3) which had been suggested for the actual dam, but placed in symmetrical, rectangular-, and sloping-sided valleys respectively. They were made to provide information as to the best methods of experimentation and also to indicate the general suitability of cast rubber for the purpose in mind.

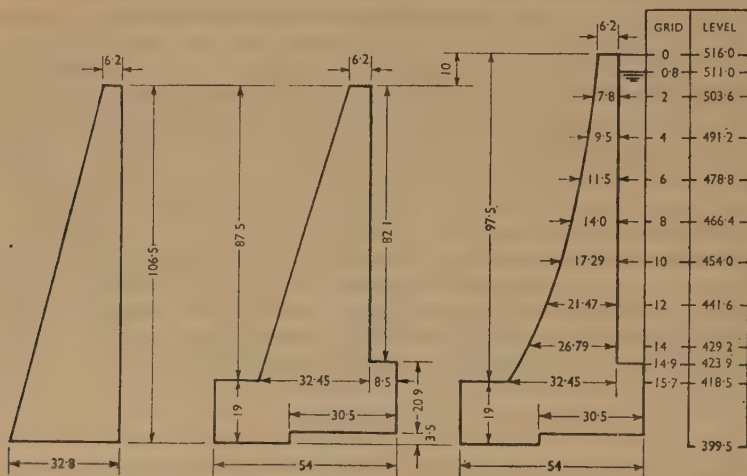


FIG. 3.—DESIGN D.1 FIG. 4.—DESIGN D.2.

FIG. 5.—DESIGN D.3

No accurate model of the dam of profile D.1 was made, since approximate calculations indicated that it was likely to be rather highly stressed, and it was replaced by a heavier trapezoidal section D.2 (Fig. 4). While approximate relaxation calculations were being made on D.2 the 1 : 253 scale model shown in Fig. 9, facing p. 209, was moulded and tested. Tumbler calculations indicated that this profile was unnecessarily heavy and a modified design was produced. This, the final design, D.3 (Fig. 5) contained about the same total amount of concrete as D.2 but material saved by replacing the straight air face by a cubic curve was used to raise the crest of the dam by 10 m and so increase the volume of the reservoir by about 40%. A model of D.3 (Fig. 10, facing p. 209) was made to the same scale as that of D.2 and systematically studied. The experiments provided useful information on the behaviour of the structure and detailed comparison of the results obtained from them with those of relaxation calculations gave confidence in the estimates of stresses likely to occur in the dam.

Construction of moulds

The moulds for the two pilot models were made of 18-gauge steel and presented no difficulty since both surfaces of the dam were developable.

The mould for D.2 was more complicated. Like the pilot models, the back and

front were both developable surfaces and were formed from flat sheets, but the valley, no longer idealized, represented the actual Dokan site. It was originally intended that the model should be inverted for a proposed experimental determination of gravity displacements, although this was later found to be unnecessary. For this reason the base was contained in a brass box of 16-gauge sheet, for it was understood from the moulders that rubber could be made to bond with this material but would come away cleanly from the steel forming the mould for the main structure. The mould is shown in Figs 11a and b, facing p. 224.

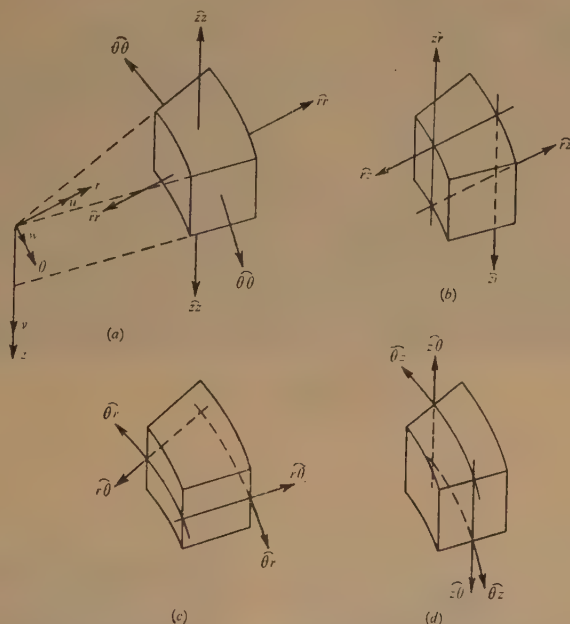


FIG. 6.—NORMAL STRESSES (a) AND COMPLEMENTARY SHEAR STRESSES (b, c, AND d)

The mould for D.3 was, with minor modifications, similar to that for D.2, but since the front of the dam was not a developable surface a wooden former (Fig. 12, facing p. 225) had to be made for the front plate (Fig. 13). This former and part of the mould were made in the Department of Civil Engineering, Imperial College, but the air face was formed in sheet metal and the mould completed by the Kensington Sheet Metal Co. Ltd.

Physical properties of rubber

Test specimens of rubber, cast at the same time as the models, were used to determine the physical properties of the material. These specimens, approximately $\frac{3}{4}$ in. $\times$ $\frac{3}{4}$ in. in cross-section and about 12 in. long, were cast horizontally in tinned steel moulds.

Determination of density

The specimens were weighed and their volumes found from the accurately measured dimensions. The density of all samples was 60 lb/cu. ft.

Determination of Poisson's ratio

If an elastic body is subjected to a hydrostatic pressure of intensity p , the strain in any direction is $e = (1 - 2\sigma)\frac{p}{E}$ where σ is Poisson's ratio. Test pieces were, therefore, placed in a triaxial loading machine and hydrostatic pressure of about $2E$ was applied. No strain was detectable, even under this loading, by a cathetometer and, therefore, from the strain equation above, Poisson's ratio is 0.5.

Determination of Young's modulus

Steel needles thrust through the test piece at short distances from either end formed supporting spindles. At equal distances from these, two similar needles carried slings made of thread which supported equal loads to produce pure bending

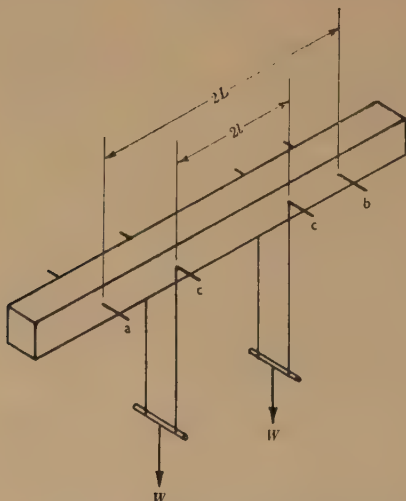


FIG. 14.—RUBBER TEST SPECIMEN

- a. Support spindle rests on knife edges
- b. Support spindle rests on smooth flats
- c. Loading spindles

over the central portion of the test piece. One end spindle was located in small V-notches made in knife-edges; the other rested on plain knife-edges which reduced frictional constraint to the minimum. The small overhang at each end was sufficient to keep the test piece approximately straight under its own weight alone; the arrangement is shown in Fig. 14. Deflexions at the centre of the test piece and at the load spindles were measured, and the mean deflexion of the centre point below the load spindles was thus obtained; the test piece was then inverted and the test repeated. If the deflexion of the centre of the test piece below the spindles is denoted by Δ , the supported length of the beam by $2L$, and the distance between load points by $2l$, Young's modulus is then:

$$E = \frac{W(L-l)l^2}{2l\Delta},$$

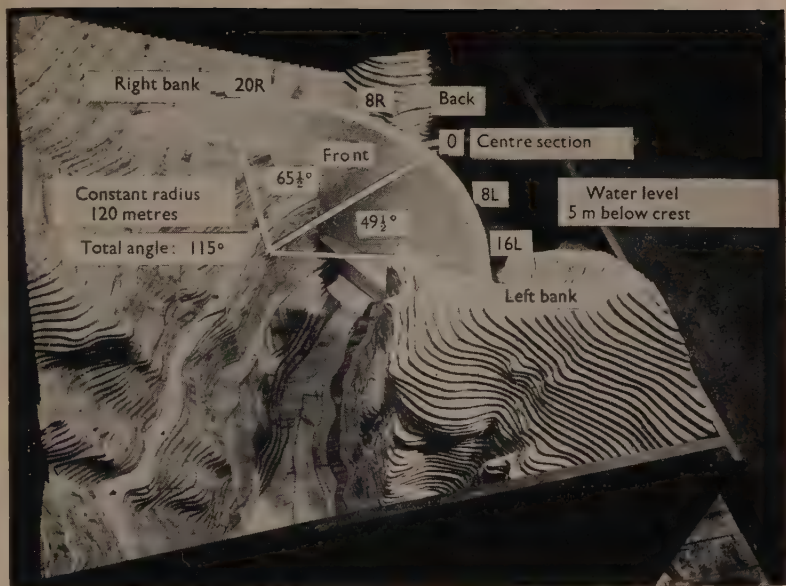


FIG. 1.—MODEL OF PRELIMINARY DESIGN



FIG. 7.—EXPERIMENTAL PILOT MODEL
P.M.1



FIG. 8.—EXPERIMENTAL PILOT MODEL
P.M.2

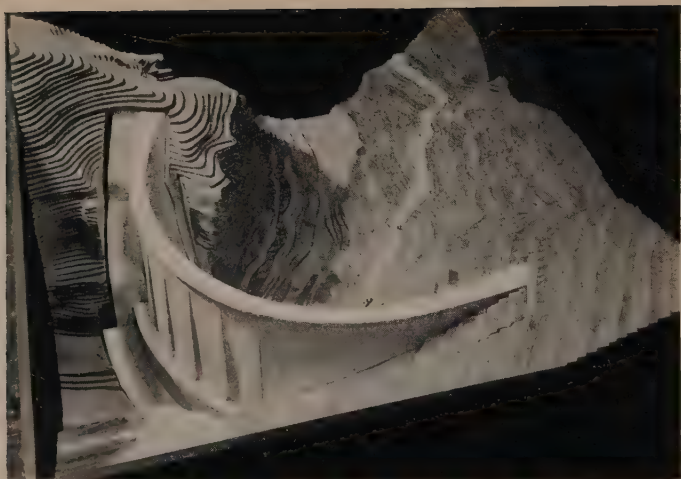


FIG. 9.—DESIGN D.2. MODEL IN VALLEY



FIG. 10a.—DESIGN D.3. RUBBER MODEL (BACK)

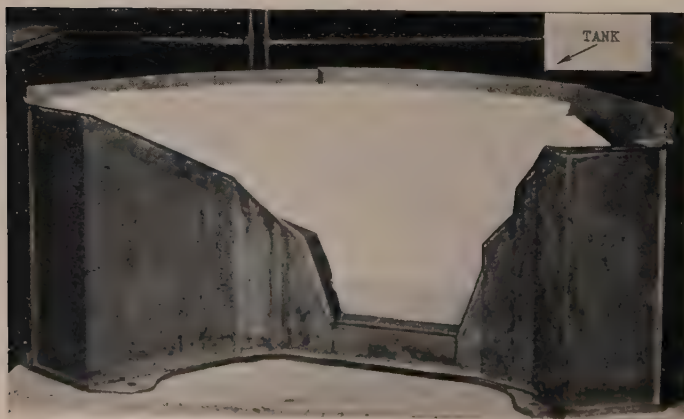


FIG. 10b.—DESIGN D.3. RUBBER MODEL (FRONT)

where W denotes the load applied at *each* load point and I the second moment of area of the cross-section of the test piece.

Typical curves obtained in this way are shown in Fig. 15 from which it is seen that the relation between load and deflexion is reasonably linear.

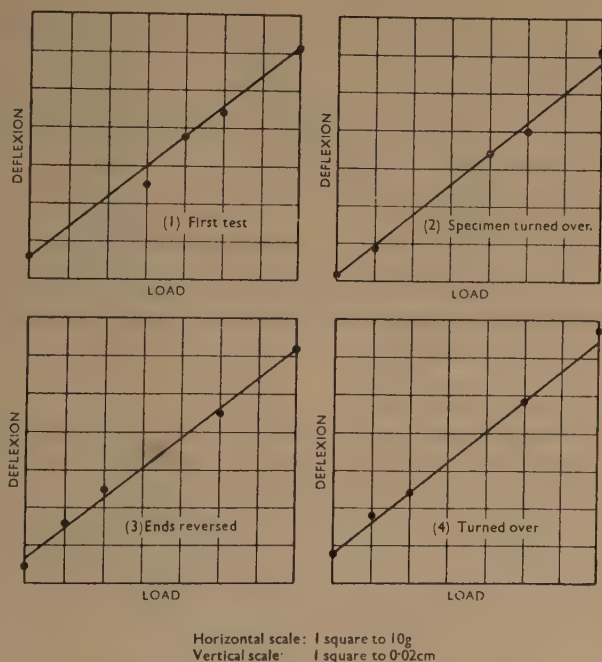


FIG. 15.—LOAD/DEFLEXION CURVES FOR RUBBER

Table 1 gives the values of Young's modulus thus determined for the material of the models.

TABLE 1.—VALUES OF YOUNG'S MODULUS FOR RUBBER

Model	Value of E : lb/sq. in.
P.M.1	115
P.M.2	115
D.2	200
D.3	109

Measurement of displacements due to water pressure

The two pilot models were mounted in small tanks and embedded in plaster of Paris. The upstream portion of the tank was filled with water and the resulting

displacements were measured by a cathetometer mounted above the model on an independent staging.

For the larger models, D.2 and D.3, the tank was replaced by a casing of tin plate which was fastened to the base-plate, roughly conforming to the curve of the dam; this avoided excessive waste of water and also reduced the time taken to load and unload the model. To prevent the casing imposing any restraint on the model it was connected to it only by sealing strips of thin rubber attached along the varying level of the ground line.

Models D.2 and D.3 were set in cement concrete on a concrete basement floor. The importance of a really solid foundation was emphasized after slight movements of the floor upon which D.2 was originally placed caused inconsistencies in the results of early experiments.

All measurements were made as before by cathetometers carried on independent staging. After each alteration in loading the models needed a certain time to assume their final configurations; 5 min was sufficient and this interval was allowed before any measurements were made. Repeat readings of displacements were taken at each point. In the initial stages of the work each reading entailed a separate loading, but later it was found preferable to take observations at a number of points after each change of load.

Determination of gravity displacements

When the investigation was begun it was thought necessary to measure the displacements due to gravity and although the method of construction of the dam removed this need, it is worth recording the technique used.

The method, which at first seemed to offer a good chance of success, was to mount the model on a turntable so that it could be inverted; it would then be subjected to gravity loads opposite in direction to those acting normally and the displacements produced by inversion would be twice those due to gravity and of opposite sign. This method was tried crudely on model P.M.1 and reasonable results were obtained, but it was then discarded in favour of a second and more convenient method, which consisted in immersing the model in water and measuring the consequent displacements. It is of interest to record that after this method had been used it was proposed independently.⁸

Under gravity alone any point on the model will displace upstream by an amount $u_g = k\rho_m/E_m$, where ρ denotes specific gravity, E Young's modulus, and k a constant. When the model is immersed this displacement is reduced to $k(\rho_m - 1)/E_m$, so that the displacement measured on immersion is $u_m = -k/E_m = -u_g/\rho_m$. The corresponding displacement of the prototype under gravity is then $-u_m n^2 \rho_p E_m/E_p$, the suffixes m and p denoting model and prototype respectively.

RESULTS OF MODEL EXPERIMENTS

Experiments on model P.M.1

P.M.1 (Fig. 7) was of trapezoidal section in a rectangular valley and was used partly to explore the possibilities of cast rubber and partly to check the results of a tumbler analysis.

The radial displacements at the centre section of the model are given in Table 2; gravity and pressure displacements were measured separately, but the tumbler values also given in the Table are due to pressure only, with $E = 115 \text{ lb./sq. in.}$ and $\sigma = 0.5$.

TABLE 2.—RADIAL DISPLACEMENTS AT CENTRE OF P.M.1

Distance from base : inches	Measured: inch			Calculated : inch
	Pressure	Gravity	Net	Pressure
10	—0.152	0.021	—0.131	—0.149
9	—0.148	0.017	—0.131	—0.151
8	—0.145	0.013	—0.132	—0.148
7	—0.136	0.0095	—0.126	—0.138
6	—0.125	0.007	—0.118	—0.125
5	—0.105	0.0055	—0.100	—0.109
4	—0.092	0.003	—0.089	—0.091
3	—0.069	0.002	—0.067	—0.070
2	—0.046	0	—0.046	—0.047
1	—0.030	0	—0.030	—0.031
0	—0.009	0	—0.009	—0.014

The close agreement between measured and calculated deflexions due to water pressure is shown in Fig. 16.

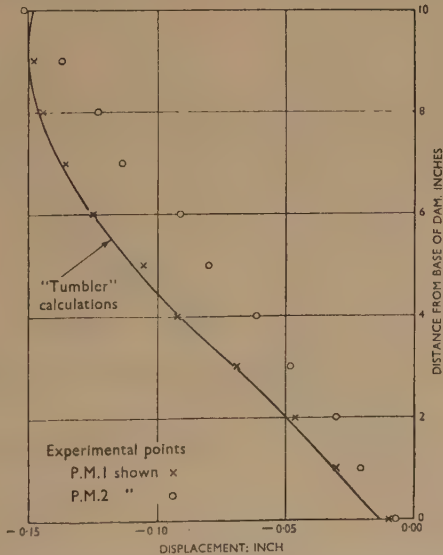


FIG. 16.—DEFLEXION OF CENTRE SECTION CAUSED BY WATER PRESSURE ON P.M.1 AND P.M.2

Experiments on model P.M.2

This model had the same profile as P.M.1, but the valley had sloping sides as shown in Fig. 8. Experimental results are given in Table 3. It should be remembered that tumbler calculations are the same as for P.M.1, since they are independent of adjoining sections and therefore of the valley shape.

The experimental results plotted in Fig. 16 for comparison with those obtained for P.M.1 show the effect of alteration in valley shape.

TABLE 3.—RADIAL DISPLACEMENTS AT CENTRE OF P.M.2

Distance from base: inches	Measured displacements: inch		
	Pressure	Gravity	Net
10	-0.152	0.019	-0.133
9	-0.137	0.016	-0.121
8	-0.123	0.015	-0.108
7	-0.114	0.014	-0.100
6	-0.092	0.012	-0.080
5	-0.080	0.010	-0.070
4	-0.061	0.007	-0.054
3	-0.048	0.005	-0.043
2	-0.030	0.004	-0.026
1	-0.021	0.002	-0.019
0	-0.008	0	-0.008

Experiments on model D.2

Radial displacements due to water pressure were measured on two sections near the centre of the dam designated as 77 and 66 (see Fig. 17). These are shown in Table 4, scaled to full size.

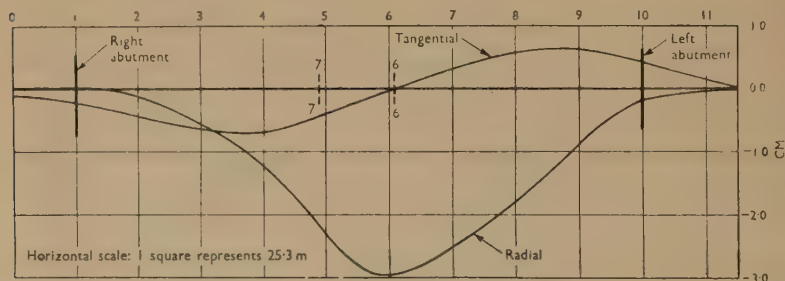


FIG. 17.—DESIGN D.2. RADIAL DISPLACEMENTS AT CREST OF FULL-SCALE DAM, DEDUCED FROM MODEL

TABLE 4.—FULL-SCALE RADIAL DISPLACEMENTS IN D.2 ON SECTIONS 77 AND 66

Reference point		Displacements due to water pressure: cm	
		Section 77	Section 66
Crest	1.0	-2.4	-2.98
	0.9	-2.25	-2.74
	0.8	-2.09	-2.60
	0.7	-1.92	-2.44
	0.6	-1.70	-2.18
	0.5	-1.45	-1.88
	0.4	-1.24	-1.62
	0.3	-0.96	-1.33
	0.2	-0.72	-1.07
	0.1	-0.51	-0.79
Sill	0	-0.36	-0.525

Radial and tangential displacements around the crest of the dam also scaled to full size are given in Table 5 and Fig. 17. These were measured at chord distances along the crest at intervals of 10 cm on the model, corresponding to 25.3 m on the prototype, starting at the right abutment (station 1 on Fig. 17).

The tumbler calculations and the experimental results for this design differed to an extent which indicated that a complete analysis would be desirable for the final design D.3.

Experiments on model D.3 (final design)

All displacements measured on this model, unless otherwise stated, were for a top-water level of 511 m above datum, i.e., 5 m below the dam crest. For measuring displacements of points on the front of the dam, cathetometers were sighted directly on targets marked in Indian ink. Radial and tangential displacements were read

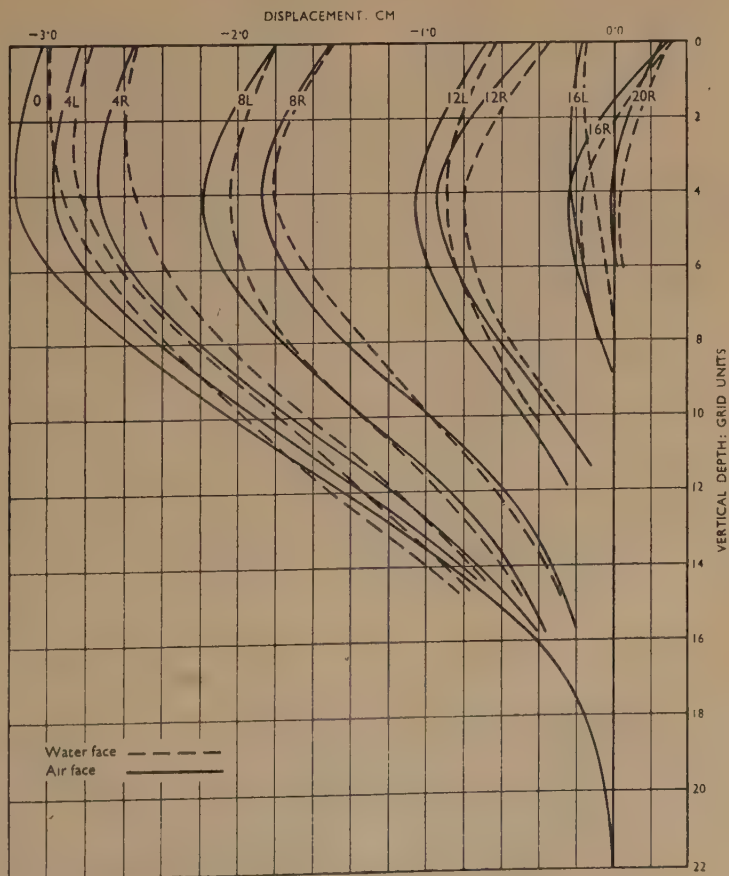


FIG. 18.- DESIGN D.3. RADIAL DISPLACEMENTS AT VARIOUS VERTICAL SECTIONS, DEDUCED FROM MODEL

TABLE 5.—FULL-SCALE RADIAL AND TANGENTIAL DISPLACEMENTS IN D.2 CREST

Station	Displacements: cm		Remarks
	Radial	Tangential	
Right bank	0	0	40 m right of 0
0	0	-0.121	
1	0.038	-0.250	Right abutment
2	-0.121	-0.440	
3	-0.594	-0.626	
4	-1.210	-0.630	
5	-2.380	-0.377	Very close to Section 77
6	-2.980	-0.064	Very close to Section 66
7	-2.480	0.288	
8	-1.840	0.595	
9	0.910	0.634	
10	-0.173	0.416	Left abutment
11	-0.051	0.121	
Left bank	0	0	40 m left of 10

TABLE 6.—FULL-SCALE RADIAL DISPLACEMENTS: CM

(The upper figure in each case relates to the water face, the lower figure to the air face)

Grid	20R	16R	12R	8R	4R	0	4L	8L	12L	16L
0	0.28 0.26	0.32 0.29	-0.36 -0.42	-1.50 -1.52	-2.53 -2.55	-3.00 -3.03	-2.77 -2.82	-1.82 -1.80	-0.63 -0.69	-0.14 -0.18
2	0.15 0.08	0.02 -0.06	-0.62 -0.76	-1.73 -1.76	-2.61 -2.68	-2.98 -3.14	-2.86 -2.94	-1.97 -2.05	-0.84 -0.91	-0.16 -0.23
4	0.04 -0.01	-0.17 -0.24	-0.80 -0.95	-0.81 -1.87	-2.53 -2.74	-2.94 -3.17	-2.83 -2.97	-2.04 -2.18	-0.85 -1.06	-0.11 -0.23
6	0.03 0.01	-0.17 -0.21	-0.74 -0.85	-1.62 -1.75	-2.38 -2.56	-2.66 -2.98	-2.61 -2.80	-1.97 -2.07	-0.85 -0.99	0 -0.16
8		-0.09 -0.06	-0.56 -0.62	-1.33 -1.43	-2.07 -2.21	-2.38 -2.55	-2.23 -2.41	-1.73 -1.76	-0.67 -0.78	0 -0.09
8.8		— -0.02								
10			-0.26 -0.31	— -0.98	— -1.71	— -2.03	— -1.89	— -1.33	-0.41 -0.49	
		at 11.4 {	— -0.12						— -0.25	at 11.9
12				-0.63 -0.60	-1.14 -1.18	-1.51 -1.42	-1.32 -1.33	-0.93 -0.87		
14				-0.35 -0.33	-0.75 -0.72	-0.99 -0.86	-0.89 -0.82	-0.59 -0.55		
15.7								-0.38		

from directly above with the cathetometer bar horizontal; vertical displacements were read from a station downstream with the bar vertical and the telescope horizontal.

The vertical water face presented some difficulty, for it was masked by the tank and submerged when this was full. For measurement of radial and tangential displacements on this face small pins were driven into the model at reference points and sights were taken on the pins as close as possible to the dam face. To avoid refraction errors, care was taken to ensure that the line of sight was accurately vertical by

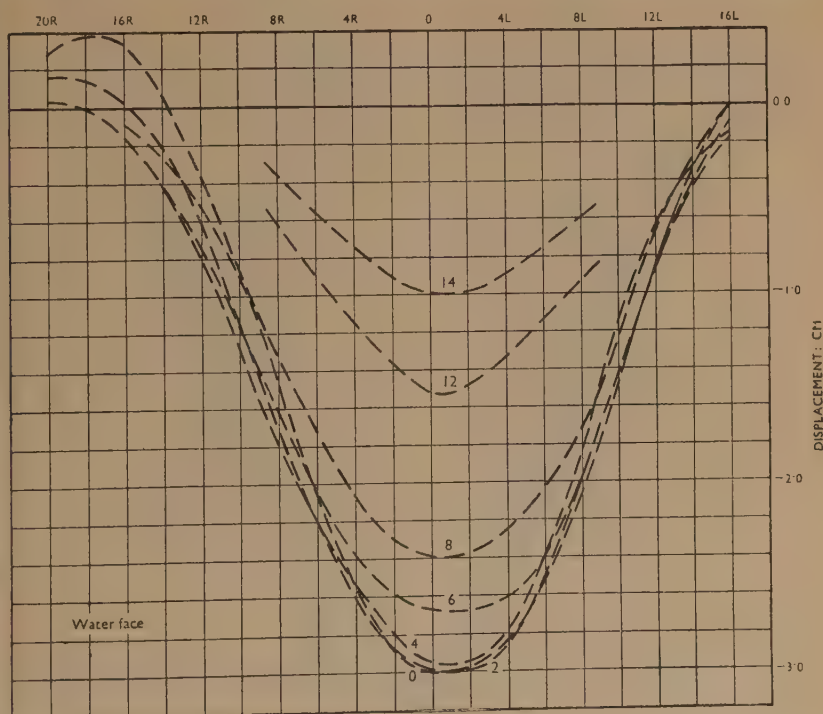


FIG. 19.—DESIGN D.3. RADIAL DISPLACEMENTS AT VARIOUS GRID LEVELS, DEDUCED FROM MODEL. CURVE NUMBERS ARE DISTANCES IN GRID UNITS FROM THE TOP

observing points at the bottom of the empty tank which were known to remain fixed and adjusting the instrument so that there was no change in reading when they were viewed through the water in the full tank. For the measurement of vertical displacements on the water face a small square of thin copper foil was secured to the dam face by the reference pin and a fine wire attached to the square passed over a pulley about 4 ft vertically above the crest; a small weight on the free end of the wire kept it taut. The cathetometer was placed with its bar vertical at a position downstream and the telescope sighted on a target on the wire slightly above the crest of the dam. The model was marked with a grid representing 6.2 m spacing on full scale, this unit being used in the relaxation. The intervals adopted for

experimental measurement were 4 grid units horizontally and 2 grid units vertically, measured on the water face; corresponding air-face points were on the same radial sections of the dam. The vertical zero grid line was not quite the same as for relaxation calculation, but was chosen arbitrarily at $49\frac{1}{2}^\circ$ from the left and $65\frac{1}{2}^\circ$ from the right bank.

In earlier models every measurement had been made with a separate filling and emptying of the tank and the first measurements on D.3 were made in this way; a complete set of air-face displacements was thus obtained in about 3 weeks. It was

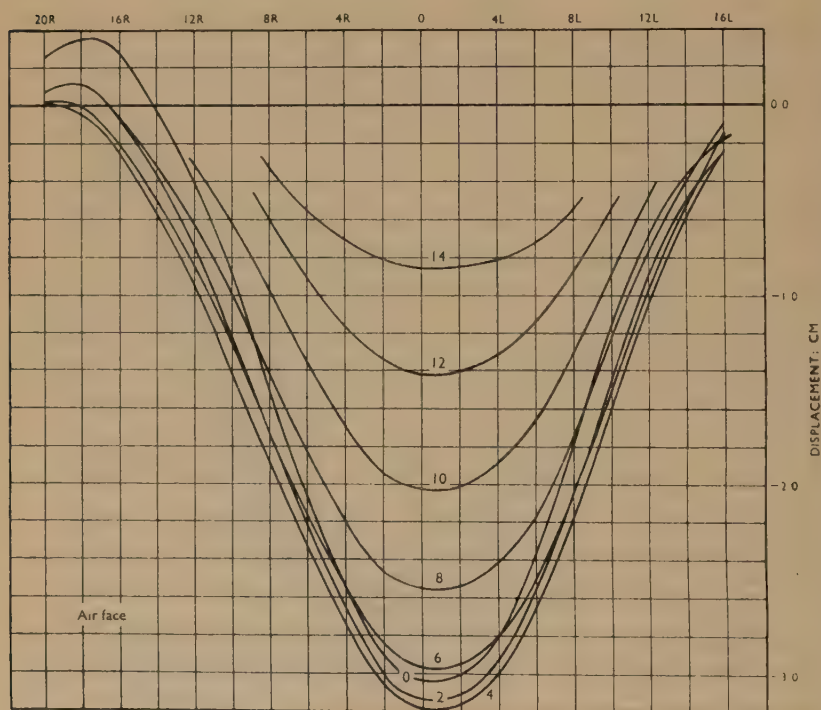


FIG. 20.—DESIGN D.3. RADIAL DISPLACEMENTS AT VARIOUS GRID LEVELS, DEDUCED FROM MODEL. CURVE NUMBERS ARE DISTANCES IN GRID UNITS FROM THE TOP

then found that although the control specimen still gave the original value of Young's modulus, 109 lb/sq. in., the values of the radial displacements were slowly but sensibly decreasing. They appeared, however, to be reduced in very much the same proportion at different points so that only the scale and not the shape of the curves was affected; this could be accounted for by a progressive increase in the effective value of Young's modulus. Since the full-scale displacements and the stresses deduced from them depend on the product of this modulus and the model displacement, a technique was devised for measuring a large number of displacements, both upstream and downstream, with one filling and emptying of the tank. In this way a complete set was obtained in a short time during which Young's modulus was

sensibly constant and judged to be 120 lb/sq. in. It was then only necessary to multiply these displacements by $n^2 E_m/E_p = 3.81$ to obtain the full-scale displacements given in Tables 6, 7, and 8 and in Figs 18 to 24. It might be emphasized at this point that the rubber model under water load gave displacements as large as a quarter the size of those to be expected on the full-sized dam. These curves were then available as guides for the complete relaxation analysis.

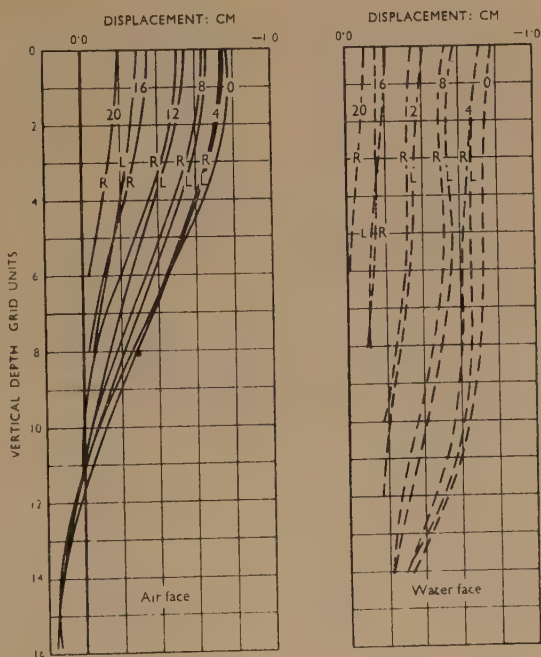


FIG. 21.—DESIGN D.3. VERTICAL DISPLACEMENTS ON VARIOUS VERTICAL SECTIONS, DEDUCED FROM MODEL

One fact emerged clearly from them: that there was to all intents a vertical plane of elastic symmetry for the displacements at approximately 48° from the left and 67° from the right abutment, even in the upper part of the dam where geometrical symmetry ceased to hold.

One further set of curves obtained at a later date (Fig. 25) shows the radial displacements at the centre section for different water levels in the dam. These are included to show the remarkable increase in the deflexions of the dam at the crest caused by the top few metres of water in the reservoir.

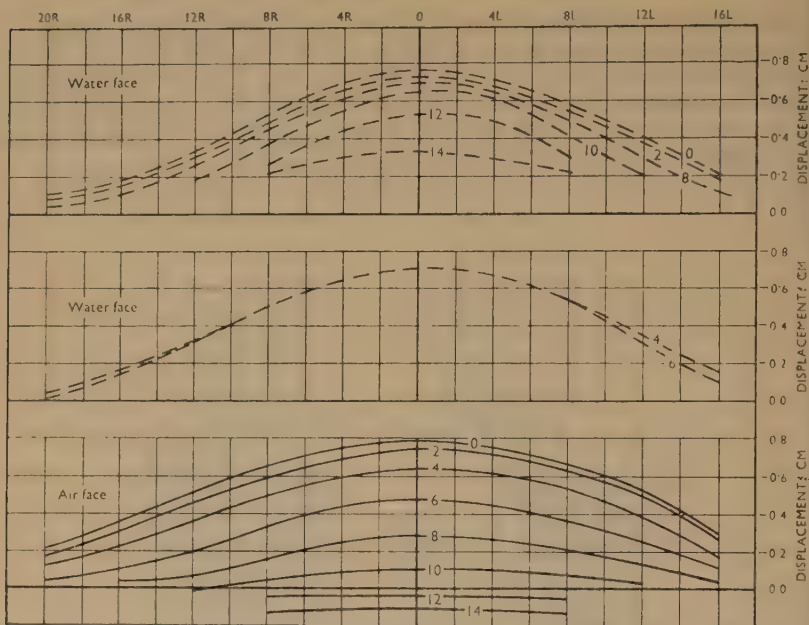


FIG. 22.—DESIGN D.3. VERTICAL DISPLACEMENTS AT VARIOUS GRID LEVELS, DEDUCED FROM MODEL. CURVE NUMBERS ARE DISTANCES IN GRID UNITS FROM THE TOP

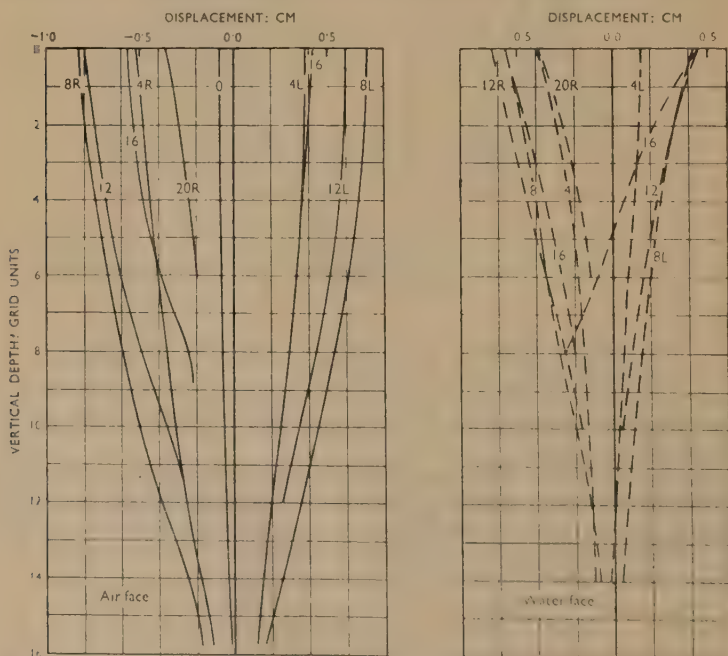


FIG. 23.—DESIGN D.3. TANGENTIAL DISPLACEMENTS AT VARIOUS VERTICAL SECTIONS, DEDUCED FROM MODEL.

TABLE 7.—FULL-SCALE VERTICAL DISPLACEMENTS: CM

(The upper figure in each case relates to the water face, the lower figure to the air face)

Grid	20R	16R	12R	8R	4R	0	4L	8L	12L	16L
0	-0.10 -0.20	-0.16 -0.35	-0.35 -0.51	-0.52 -0.65	-0.71 -0.75	-0.77 -0.77	-0.71 -0.76	-0.57 -0.66	-0.41 -0.54	-0.21 -0.30
2	-0.07 -0.16	-0.16 -0.30	-0.31 -0.46	-0.49 -0.59	-0.67 -0.68	-0.74 -0.76	-0.67 -0.69	-0.53 -0.62	-0.37 -0.49	-0.18 -0.27
4	-0.04 -0.11	-0.16 -0.21	-0.32 -0.35	-0.51 -0.50	-0.64 -0.60	-0.72 -0.63	-0.66 -0.61	-0.55 -0.53	-0.35 -0.38	-0.16 -0.19
6	— -0.03	-0.14 -0.10	-0.30 -0.20	-0.50 -0.34	-0.62 -0.46	-0.72 -0.44	-0.66 -0.44	-0.54 -0.37	-0.32 -0.25	-0.11 -0.13
8		-0.10 -0.03	-0.26 -0.06	-0.45 -0.16	-0.60 -0.26	-0.70 -0.28	-0.65 -0.27	-0.49 -0.21	-0.28 -0.14	-0.11 -0.05
10			-0.18 0.01	-0.37 -0.04	-0.56 -0.08	-0.64 -0.08	-0.62 -0.11	-0.41 -0.08	-0.20 -0.03	
		at 11.4 {	— 0.02						-0.17 —	at 11.9
12				-0.26 0.04	-0.45 0.04	-0.52 0.05	-0.50 0.02	-0.31 0.05		
14				-0.22 0.13	-0.30 0.09	-0.33 0.10	-0.30 —	-0.22 0.13		
15.7				— 0.13	— 0.14	— 0.18				

RESULTS OF RELAXATION TUMBLER ANALYSES

The theoretical basis of the relaxation analysis is given in the Appendix; the results were presented in a number of diagrams some of which related to the earlier designs which were of a preliminary or exploratory character. It is unnecessary to reproduce these and attention is concentrated on the work done in connexion with the final design D.3.

Water pressure and gravity loading

Figs 26a and b, Plate 1, show the displacements of all points in the section obtained from a tumbler analysis based on the central section of the dam due to water pressure alone. Each displacement was calculated to one more significant figure than is shown. Figs 26c to f give the radial, hoop, vertical, and shear stresses to the nearest 5 lb/sq. in. deduced from these displacements. Fig. 27, Plate 1, gives the tumbler $\widehat{zz}$ stresses when gravity forces alone act on the same section; Fig. 28 gives the tumbler $\widehat{zz}$ stresses due to gravity forces alone acting on another section of the dam for comparison. The radial $\widehat{rr}$ stresses and the shear $\widehat{zr}$ stresses were also computed for both these sections, but they are not quoted since nowhere did they

TABLE 8.—FULL-SCALE TANGENTIAL DISPLACEMENTS: CM

(The upper figure in each case relates to the water face, the lower figure to the air face)

Grid	20R	16R	12R	8R	4R	0	4L	8L	12L	16L
0	-0.39 -0.35	-0.55 -0.56	-0.63 -0.79	-0.55 -0.82	-0.40 -0.51	— -0.069	0.15 0.39	0.42 0.71	0.45 0.60	0.45 0.43
2	-0.26 -0.29	-0.47 -0.53	-0.54 -0.74	-0.47 -0.78	-0.29 -0.48	— -0.065	0.14 0.39	0.33 0.69	0.34 0.59	0.23 0.40
4	-0.17 -0.23	-0.35 -0.48	-0.45 -0.68	-0.40 -0.72	-0.23 -0.45	— —	0.11 0.36	0.26 0.66	0.22 0.56	0.06 0.37
6	-0.12 -0.19	-0.27 -0.39	-0.34 -0.60	-0.34 -0.67	— -0.40	— -0.055	0.07 0.34	0.19 0.61	0.16 0.52	-0.07 0.34
8		-0.20 -0.24	-0.25 —	-0.28 -0.58	-0.15 -0.36	— —	— 0.30	0.15 0.54	0.09 0.44	-0.25 0.29
8.8		— -0.22								
10			-0.17 -0.35	— -0.50	— -0.31	— -0.048	— 0.25	— 0.46	0.04 0.36	
		at 11.4 {	— -0.27						— 0.25 }	at 11.9
12				-0.11 -0.38	-0.11 -0.25	— —	0.01 0.21	0.07 0.37		
14				-0.07 -0.23	-0.10 -0.17	— -0.024	-0.02 0.15	— 0.25		
15.7				— -0.17			— 0.12	— 0.17		

exceed 30 lb/sq. in.; in fact over most of the section they were both less than 10 lb/sq. in. It was from calculations such as this that the theoretical downstream displacements of the pilot models plotted in Fig. 16 were obtained.

Temperature stresses

The difficulties associated with calculation of stress due to changes of temperature have already been discussed, but in spite of inherent uncertainties it was obviously desirable to make as good an analysis as seemed reasonable. The first difficulty was to assess the temperature variations that might be expected in the completed dam in the light of knowledge of air and water temperatures at the site.

The air temperature records at Dokan are very similar to those at the site of the Boulder Dam, and advantage was taken of this and of knowledge of temperatures in the Boulder Dam to deduce the annual mean temperatures to be expected in the Dokan dam and the maximum variations from these means. The result of this analysis is summarized in Figs 29a and b; the temperatures given are those on the surface of the concrete and relaxation calculations were made to determine the distribution throughout the structure under mean temperature conditions on the

assumption that a stable state was reached, and this is shown in Figs 30a to c. Plate 2. Top water level and tail water level were taken to be 511 m and 413.5 m respectively. The stresses caused by maximum variations of temperature above and below the mean were calculated by the "tumbler" process, also on the assumption of a stable temperature state; the results of these calculations are given in Figs 31a to d, Plate 2.

If contraction joints are assumed to be grouted at mean temperatures, the greatest additional vertical stress from the tumbler solution at the centre of the dam due to maximum temperature conditions is 130 lb/sq. in. compression on the air face and 146 lb/sq. in. tension on the water face. The maximum additional hoop stress is

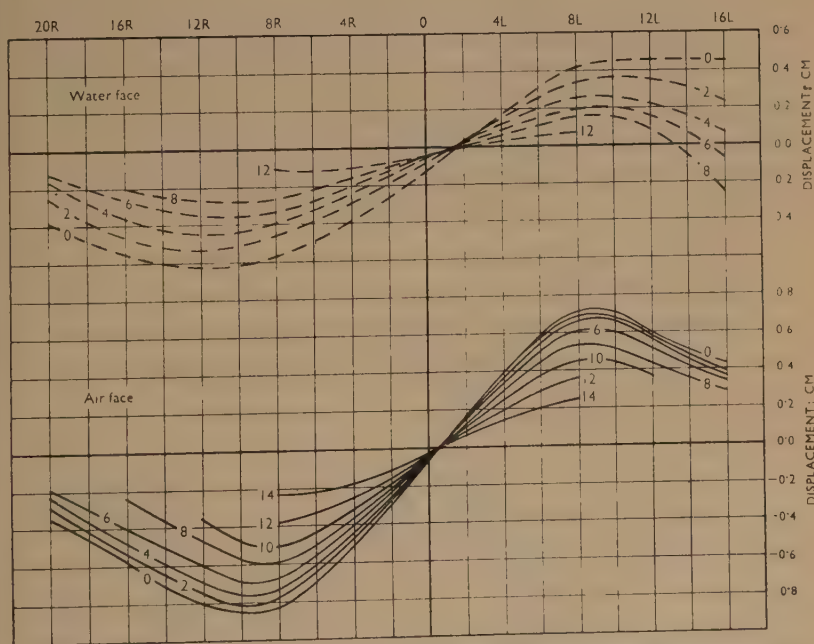


FIG. 24.—DESIGN D.3. TANGENTIAL DISPLACEMENTS AT VARIOUS GRID LEVELS, DEDUCED FROM MODEL. (CURVE NUMBERS ARE DISTANCES IN GRID UNITS FROM THE TOP)

220 lb/sq. in. compression on the air face and 29 lb/sq. in. tension on the water face. Under minimum temperature conditions the maximum additional stresses are the same in magnitude but tensions and compressions are interchanged.

If the joints are grouted at some temperature other than the mean but not too far removed from it, the additional stresses under extreme temperature conditions can be obtained by proportion; e.g., if the grouting temperatures are 5°F below the mean annual values, a rise of 30°F and a drop of 20°F on the air face must be allowed instead of an equal rise and fall of 25°F. With sufficient accuracy this gives 20% increase over the above stresses under maximum temperature conditions, and 20% decrease in those for conditions of minimum temperature.

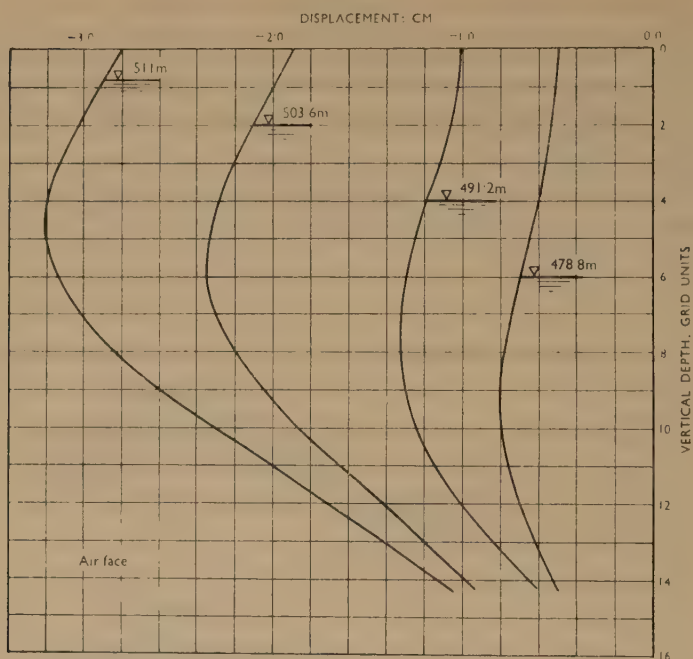


FIG. 25.—CENTRE RADIAL DISPLACEMENTS FOR VARIOUS WATER LEVELS, DEDUCED FROM MODEL

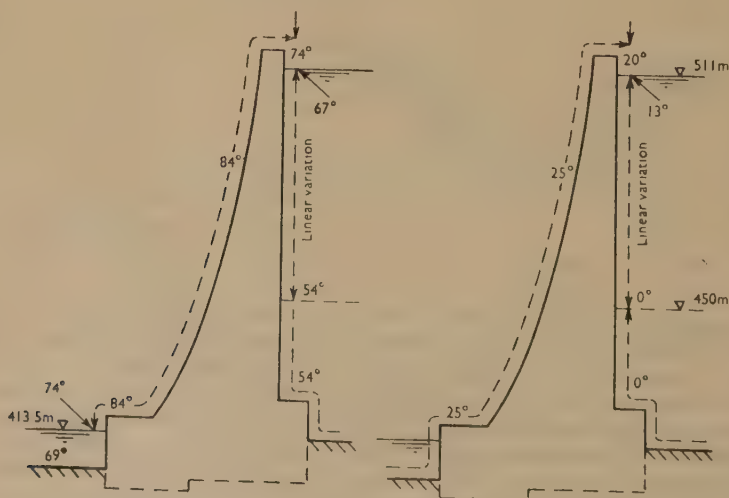


FIG. 29a.—ESTIMATED MEAN TEMPERATURE (FAHRENHEIT)

FIG. 29b.—MAXIMUM ANNUAL VARIATION

The results of the tumbler analysis are applicable with reasonable justification to the central portion of the dam but they are increasingly inaccurate towards the abutments, since the end restraints tend to relieve the stresses there. The stresses were, therefore, also calculated on the conventional assumption that the dam could be divided into a number of independent horizontal *encastré* arches, and the abutment stresses thus found are considered to be better approximations than those derived from the tumbler solution.

RESULTS OF COMPLETE RELAXATION ANALYSIS

Water-pressure stresses

The results of the complete three-dimensional relaxation analysis of the dam under water pressure are given in a series of diagrams as follows: *solely in order to make these diagrams of convenient size the dam has been developed, so that lines which are circular arcs appear as straight lines, and different scales have been used in the three co-ordinate directions.* Fig. 32, Plate 3, represents one half of the dam and part of the adjoining rock and clearly indicates the profile of the centre section. This diagram is intended to form a key for locating the mesh lines and nodes and to convey the extent of the region investigated which includes the dam itself and a thick fringe of adjoining foundation and abutment. For the sake of clarity the third set of mesh surfaces (the cylinders $r = \text{constant}$) has not been completed in the diagram.

TABLE 9.—COMBINED WATER AND GRAVITY AND TEMPERATURE LOADING
Comparison of central stresses at maximum and minimum temperatures with those at grouting temperatures.
Grouting temperatures assumed 5°F below mean temperatures.
All stresses in lb/sq. in., tension positive.

	Grid level	Vertical stress			Hoop stress		
		At grouting temperature	At maximum temperature	At minimum temperature	At grouting temperature	At maximum temperature	At minimum temperature
Water Face	0	0	0	0	-290	-290	-290
	2	-110	-165	-70	-380	-375	-380
	4	-240	-225	-250	-460	-440	-470
	6	-335	-245	-395	-480	-430	-515
	8	-340	-205	-430	-450	-370	-505
	10	-280	-110	-390	-380	-270	-450
	12	-160	15	-275	-280	-175	-350
	14	-60	115	-180	-170	-80	-230
Air Face	0	0	0	0	-190	-290	-130
	2	20	70	-10	-200	-330	-110
	4	100	75	115	-200	-380	-80
	6	120	0	200	-180	-385	-45
	8	100	-30	190	-140	-360	0
	10	30	-120	130	-90	-320	60
	12	-140	-300	-20	-50	-290	110
	14	-300	-410	-230	-20	-285	155
	sill	-330	-380	-300	0	-275	185

Figs 33a to c, Plate 3, record the component displacements at each node measured in hundredths of a centimetre. In the computations all these displacements were evaluated to one more significant figure than given here.

Figs 34a to d, Plate 4, record the six components of stress at each node to the nearest 10 lb/sq. in. From these diagrams and from the solution for gravity stresses given in Fig. 27, Plate 1, the total stresses at any point in the dam can be found by superposition. Thermal stresses at this centre section of the structure as found by a tumbler analysis and given in Figs 31a to d, Plate 3, can also be added to find the combined effect of pressure, gravity, and temperature changes at that critical section. As already explained, the tumbler solution for thermal stresses is not applicable to the abutment sections and their values have been calculated by more conventional methods.

The vertical ($\widehat{zz}$) and hoop ($\widehat{\theta\theta}$) stresses on the water and air faces at the centre section of the dam are given in Table 9 for combined water and gravity loading at normal or grouting temperatures as well as at maximum and minimum temperature conditions in the structure.

Table 10 gives the same information as calculated for the abutments, allowing for the fillets on the air face.

TABLE 10.—COMBINED WATER AND GRAVITY AND TEMPERATURE LOADING
Stresses at left abutment at grouting temperatures, with hoop stresses at maximum and minimum temperatures.
Grouting temperatures assumed 5°F below mean temperatures.
All stresses in lb/sq. in., tension positive.

	Grid level	Vertical stress	Hoop stress		
		At grouting temperature	At grouting temperature	At maximum temperature	At minimum temperature
Water face	0	0	—80	—150	—45
	2	—70	—140	—175	—150
	4	—150	—170	—210	—190
	6	—165	—140	—195	—160
	8	—155	—70	—170	—70
	10	—150	—80	—225	—60
	12	—170	50	—220	100
	14	—100	80	—200	130
Air face	0	0	—260	—190	—295
	2	0	—305	—275	—295
	4	30	—340	—310	—320
	6	—20	—380	—335	—355
	8	—115	—275	—200	—265
	10	—120	—190	—100	—180
	12	—105	—250	—155	—230
	14	—230	—315	—280	—260

STRESS CONCENTRATIONS

It will be noted that severe stress concentrations are indicated in the various solutions near sharp re-entrant corners of the profile. It should be explicitly stated



FIG. 11a.—DESIGN D.2. MOULD FOR MODEL.

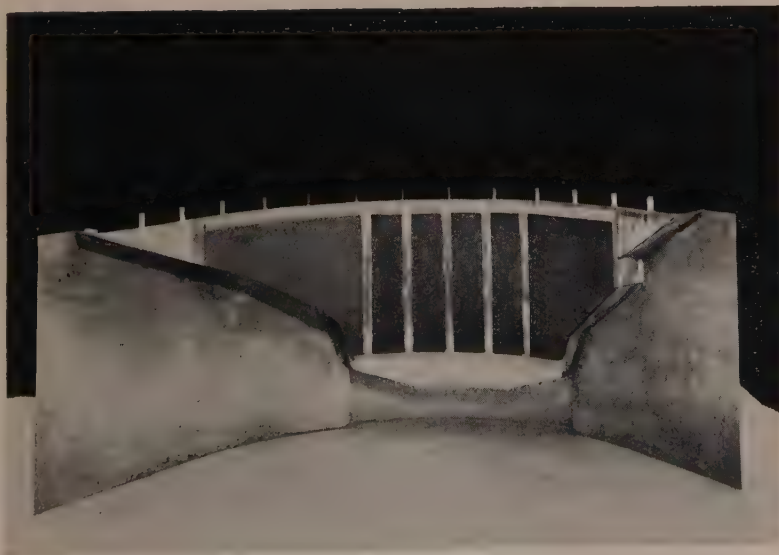


FIG. 11b.—DESIGN D.2. MOULD WITH FRONT REMOVED SHOWING LINO IN PLACE INSIDE



FIG. 12.—DESIGN D.3. WOODEN FORMER FOR FRONT
FACE OF MOULD

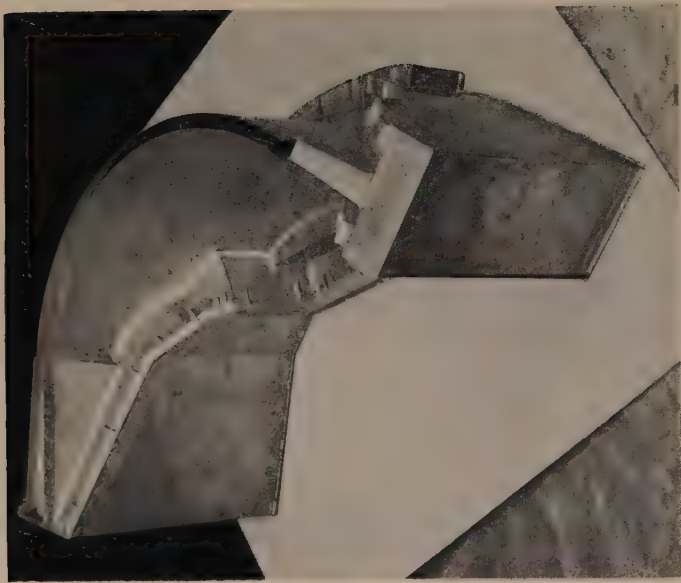


FIG. 13.—DESIGN D.3. MOULD FOR MODEL

that if they were infinitely sharp a precise mathematical solution would show infinite stresses at the actual corners and very high values in their vicinity. Investigations into other problems have shown that these concentrations are very localized and that the introduction of small corner fillets relieves them considerably. In practice it is impossible to obtain very sharp corners and the true stresses will be much smaller. Furthermore, any localized high stresses will be relieved by the natural creep of the concrete. The stresses quoted, however, are as given by the calculations and no allowance has been made either for the deliberate provision of fillets or for the effect of creep.

SUMMARY OF RESULTS

The maximum direct stresses occur on the centre section (vertical relaxation grid 0 in Fig. 2).

Assuming the dam to remain throughout at grouting temperatures, the maximum combined stresses due to gravity and water pressure are as in Table 11.

TABLE 11

Stress: lb/sq. in.	Face	Grid	Level: m
Vertical compression: 340	Water	8	466
Vertical tension: 120	Air	6	479
Hoop compression: 480	Water	6	479
Hoop tension: nil	—	—	—

Assuming that grouting takes place at 5°F below the mean temperatures, the maximum combined stresses due to gravity, water, and temperature changes are:

(a) For maximum temperature rise, as in Table 12:

TABLE 12

Stress: lb/sq. in.	Face	Grid	Level: m
Vertical compression: 410	Air	14	429
Vertical tension: 115	Water	14	429
Hoop compression: 440	Water	4	491
Hoop tension: nil	—	—	—

(b) For maximum temperature fall, as in Table 13:

TABLE 13

Stress: lb/sq. in.	Face	Grid	Level: m
Vertical compression: 430	Water	8	466
Vertical tension: 200	Air	6	479
Hoop compression: 515	Water	6	479
Hoop tension: 185	Air	sill	418.5

COMPARISON OF CALCULATED AND MEASURED DISPLACEMENTS

When deducing the full-scale displacements from the measurements on the rubber model as described on p. 217 no account was taken of the difference in Poisson's ratio between rubber and concrete. There is, in fact, no simple way to make such a correction, and it is, therefore, interesting to examine whether it is possible to account for the difference between displacements calculated by relaxation on the assumption that Poisson's ratio σ was 0.15 and those deduced from the rubber model for which Poisson's ratio was 0.5. To make an exact comparison it would be necessary either to do the calculations with Poisson's ratio 0.5, or use model material for which Poisson's ratio was 0.15. Neither of these courses was practicable, the first because of the time which would have been required, the second because no suitable model material was available.

It is possible, however, by a simple argument to obtain a qualitative idea of differences which might be expected due to this difference in Poisson's ratio. Suppose a cylinder of constant outer radius, which has a wall thickness tapering linearly from one end to zero at the other, is subjected to external radial pressure which also varies linearly along its length; calculated on thin cylinder theory, the hoop stress at every section is the same, and therefore the outside diameter will be decreased by the same amount everywhere. This is also true for a cylinder with parallel walls subjected to constant radial pressure throughout its length.

The tapered cylinder corresponds to a triangular dam under normal water pressure, on the assumption that there is no shear restraint either at the base section or at the abutments. The Dokan dam is sufficiently nearly triangular in profile for general deductions as to its behaviour to be made from the exact triangular profile and therefore, by the argument above, from the behaviour of a cylinder of constant wall thickness under uniform pressure. If such a cylinder of external radius R_1 and internal radius R_2 is subjected to an external radial pressure p , it can be shown⁹ that the decreases in the outer and inner radii are respectively:

$$\begin{aligned}\Delta_1 &= A\{(1 - \sigma)R_1^2 + (1 + \sigma)R_2^2\} \\ \Delta_2 &= A \cdot 2R_1R_2\end{aligned}$$

where

$$A = pR_1/\{E(R_1^2 - R_2^2)\}.$$

The axial strain is $2\sigma\epsilon$.

It will be observed from these relations that the alteration of the inner radius is

independent of Poisson's ratio and therefore, whatever material is used for the tube, provided E is unchanged, the inner radius should undergo the same alteration on loading. To satisfy this criterion and make, say, the maximum radial displacement on the air face deduced from the model agree with that of the calculations made with Poisson's ratio 0.15, a factor of 1.167 was needed, which is equivalent to assuming a Young's modulus of 140 instead of 120 lb/sq. in. for the rubber. This modification in Young's modulus may be interpreted as being a correction to allow for the fact that the *apparent* value of E in any direction in a body under tri-axial compressive stress is greater than the uni-axial value determined by normal tests, due to the influence of lateral strains (the Poisson effect). Such a correction can only be approximate since it will vary from point to point and be different along different axes, but the conception is useful in correlating experiments and calculations.

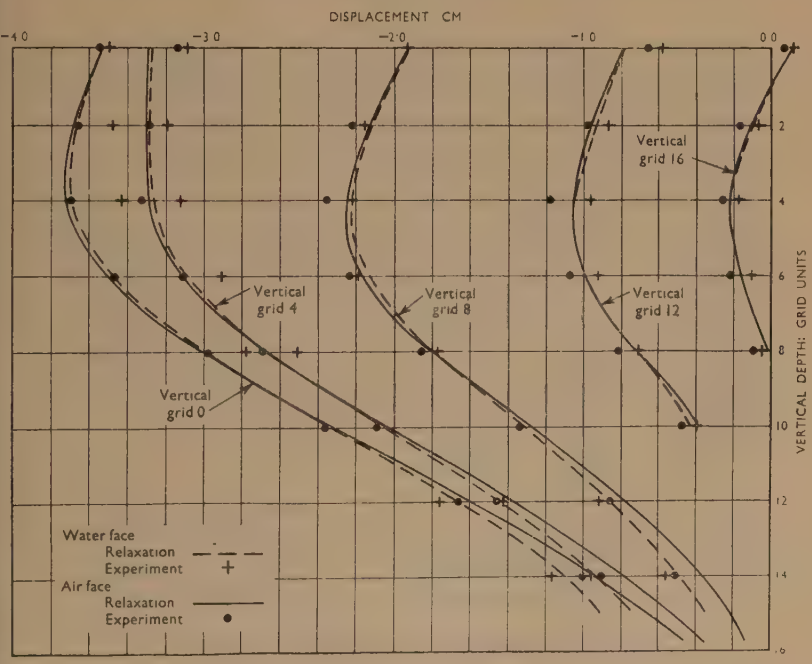


FIG. 35.—COMPARISON OF RADIAL DISPLACEMENTS u

The experimental displacements—radial, vertical, and horizontal—have been averaged for the two halves of the dam and increased by the same factor 1.167, and the results for typical sections are shown in Figs 35, 36, and 37, where they are compared with the calculated values.

Considering first the curves in Fig. 35 which relate to radial displacements, it will be noted that at the central section not only the maximum air-face displacements agree, but that there is close agreement throughout the height of the dam. Again it will be noted that the calculated values for air face and water face are close together whereas the experimental values are much farther apart: in other words, the

thickening of the section in the upper part of the dam as found experimentally was many times that determined by calculation. Using the above relations for the decreases in the two radii of the hypothetical cylinder, it follows that the increase in its thickness on loading is $A(R_1 - R_2)\{\sigma(R_1 + R_2) - (R_1 - R_2)\}$.

Provided that $0.74 < R_2/R_1 < 1$, there will be an increase in thickness for a cylinder made of concrete. For this same range of the ratio R_2/R_1 the increase in the thickness of a cylinder made of rubber would be many times that for a concrete cylinder, being about 50 times when the ratio is 0.75 and about 5 times when it is 0.9.

Fig. 36 shows that the agreement of the tangential displacements on the air face is also very close when the factor is included and the systematic differences in the water-face values may well be accounted for by the effects of foundation and abutments.

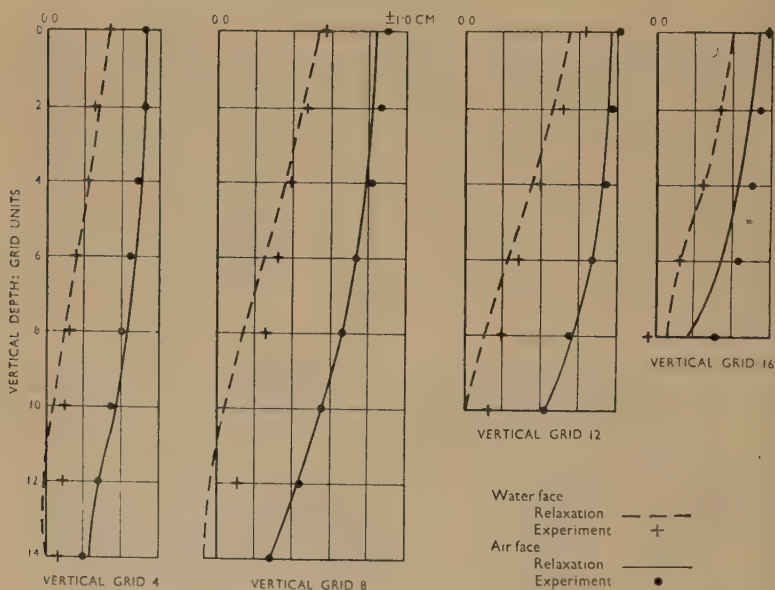


FIG. 36.—COMPARISON OF TANGENTIAL DISPLACEMENTS w

The comparison of vertical displacements in Fig. 37 shows that while the general shape of the curves is the same, those deduced from the model were greater than those calculated, and this again is in accordance with the theoretical behaviour of the hypothetical cylinders. As shown above, vertical strains, and therefore vertical displacements, are proportional to Poisson's ratio. It might then be expected that a cylinder made of rubber would have vertical displacements of about three times the magnitude of those in one made of concrete. The free movements which would occur in the cylinder are, however, prevented by the restraints due to the abutments, and the smaller disparity in the values which is shown is in accordance with actual conditions.

When the two sets of results were first compared it seemed as though the original

value of E for the rubber model had been underestimated, but this simple explanation, while accounting for radial and tangential discrepancies, did not cover the big divergence in vertical displacements. The foregoing discussion does, however,

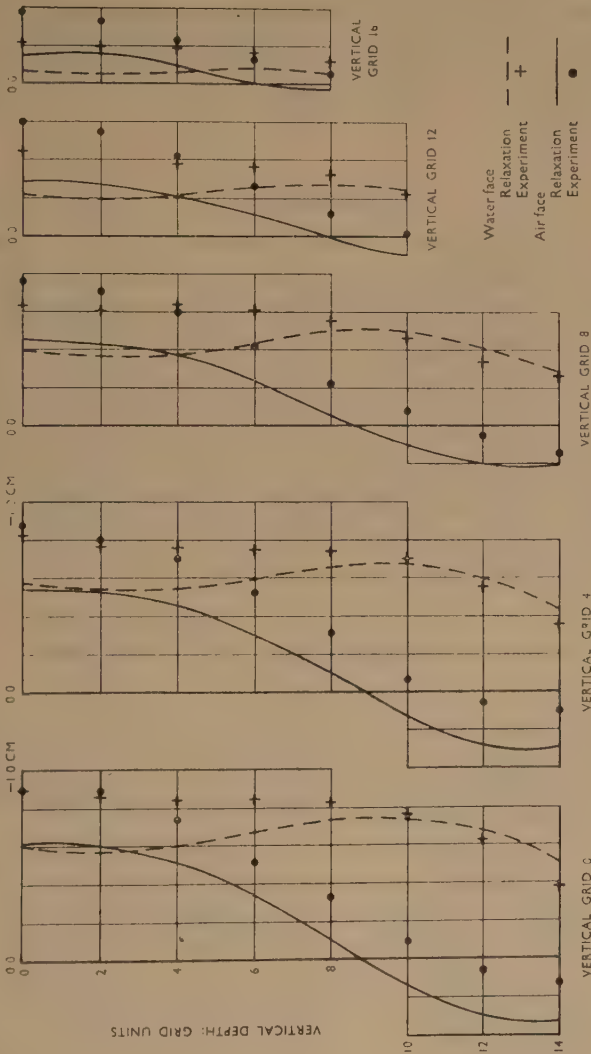


Fig. 37.—COMPARISON OF VERTICAL DISPLACEMENTS v

account for all the observed major differences between the calculated and observed displacements. To what extent it would apply in other designs can only be determined by further investigation.

ACKNOWLEDGEMENTS

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REFERENCES

1. W. P. Creager, J. D. Justin, and J. Hinds, "Engineering for Dams." Wiley, New York, 1950, vol. 2, p. 281.
2. Charles Jaeger, "Modern Trends in Arch Dam Construction and Design." *Civ. Engng & Pub. Wks Rev.*, vol. 45, p. 230 (Apr. 1950); p. 304 (May 1950); p. 379 (June 1950); p. 446 (July 1950); p. 513 (Aug. 1950); p. 522 (Sept. 1950).
3. A. Bourgin, "The Design of Dams." Pitman, London, 1953.
4. U.S. Bureau of Reclamation, "Treatise on Dams," 1950. (See Chap. 10.)
5. I. E. Houk, "Dams and Control Works." U.S. Bureau of Reclamation, 1938.
6. R. V. Southwell, "Relaxation methods in theoretical physics." O.U.P., 1946.
7. D. N. de G. Allen, "Relaxation methods." McGraw-Hill, London, 1954.
8. A. Nizery, G. Remeniéras, and N. Beaujoint, "Étude sur modèle réduit des contraintes dans les barrages." ("Study of dam loads by means of scale models.") *Ann. Ponts Chauss.*, vol. 123, p. 395 (Jul.-Aug., 1953); p. 533 (Sept.-Oct., 1953).
9. A. J. S. Pippard, "Studies in Elastic Structures." Arnold, London, 1952.
10. O. C. Zienkiewicz, "The Stress-Distribution in Gravity Dams," *J. Instn Civ. Engrs*, vol 27, p. 224 (Jan. 1947).

APPENDIX

RELAXATION ANALYSES

In this Appendix an outline only of the mathematical argument is given; a full discussion will appear elsewhere.

As explained on p. 203 the analysis was made in terms of the three component displacements at any point in the structure. Using the notation developed there, the six components of stress (Fig. 6) are given by :

$$\left. \begin{aligned} \frac{1 + \sigma}{E} \widehat{rr} &= \frac{\partial u}{\partial r} + \frac{\sigma}{1 - 2\sigma} \left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \cdot \frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} \right), \\ \frac{1 + \sigma}{E} \widehat{\theta\theta} &= \frac{1}{r} \cdot \frac{\partial w}{\partial \theta} + \frac{u}{r} + \frac{\sigma}{1 - 2\sigma} \left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \cdot \frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} \right), \\ \frac{1 + \sigma}{E} \widehat{zz} &= \frac{\partial v}{\partial z} + \frac{\sigma}{1 - 2\sigma} \left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \cdot \frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} \right), \\ \frac{1 + \sigma}{E} \widehat{\theta z} &= \frac{1}{2} \left(\frac{1}{r} \cdot \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \right), \\ \frac{1 + \sigma}{E} \widehat{zr} &= \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right), \\ \frac{1 + \sigma}{E} \widehat{r\theta} &= \frac{1}{2} \left(\frac{\partial w}{\partial r} - \frac{w}{r} + \frac{1}{r} \cdot \frac{\partial u}{\partial \theta} \right). \end{aligned} \right\} \dots \dots (1)$$

where E denotes Young's modulus and σ Poisson's ratio.

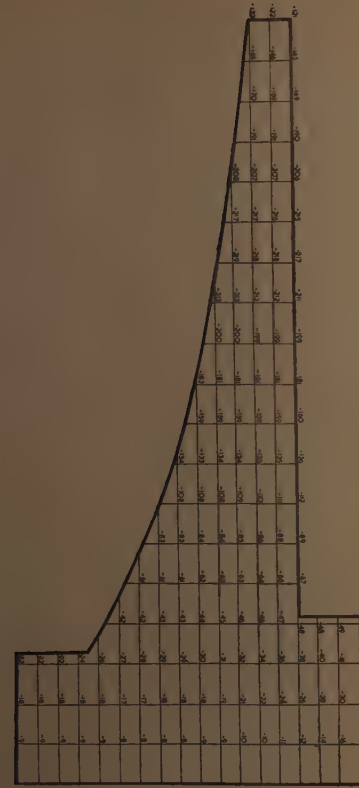


FIG. 26a.—WATER LOAD ONLY. RADIAL DISPLACEMENT u .

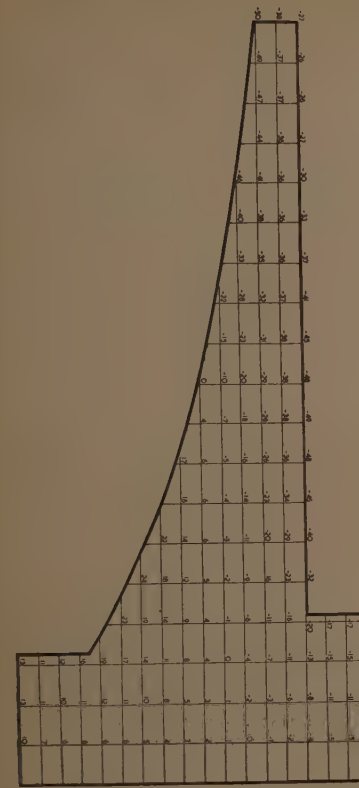


FIG. 26b.—WATER LOAD ONLY. VERTICAL DISPLACEMENT v .

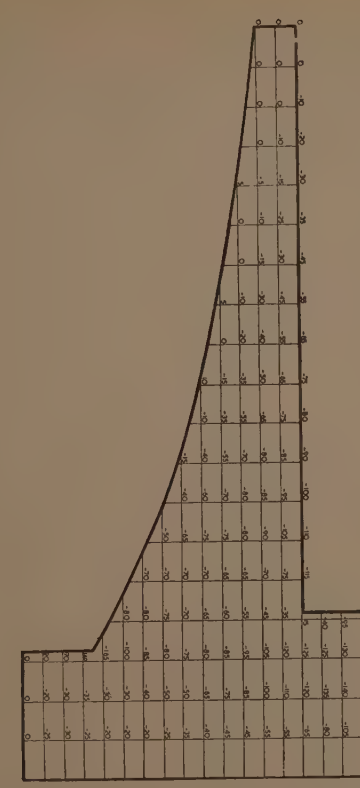


FIG. 26c.—WATER LOAD ONLY. RADIAL STRESS r_r .

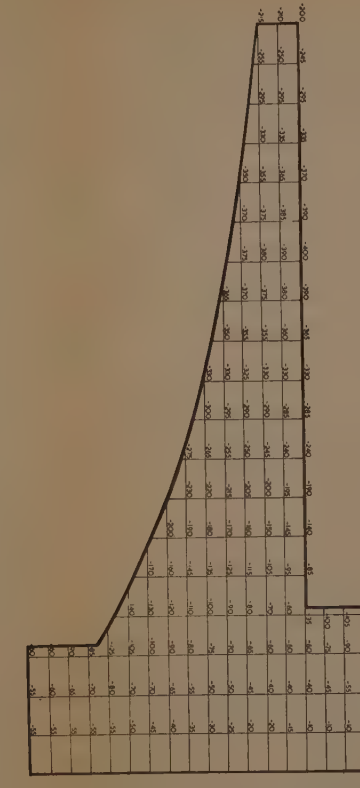


FIG. 26d.—WATER LOAD ONLY. TANGENTIAL STRESS θ_θ .

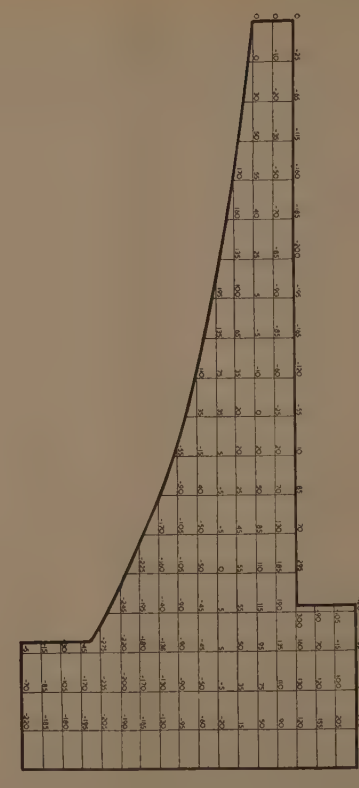


FIG. 26e.—WATER LOAD ONLY. VERTICAL STRESS z_z .

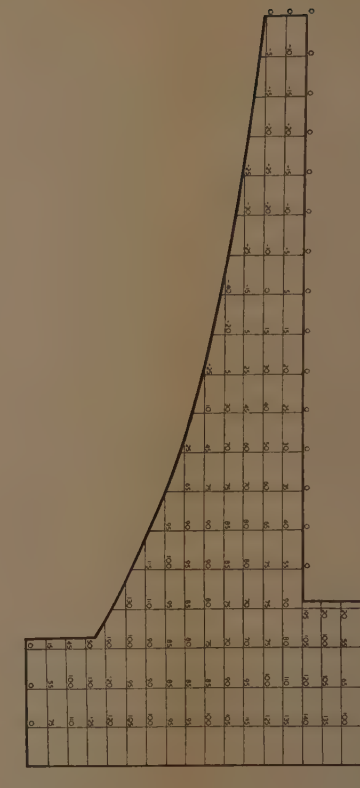


FIG. 26f.—WATER LOAD ONLY. SHEAR STRESS z_r .



FIG. 27.—GRAVITY LOAD ONLY. VERTICAL STRESS z_z .



FIG. 28.—GRAVITY LOAD ONLY. VERTICAL STRESS z_z .

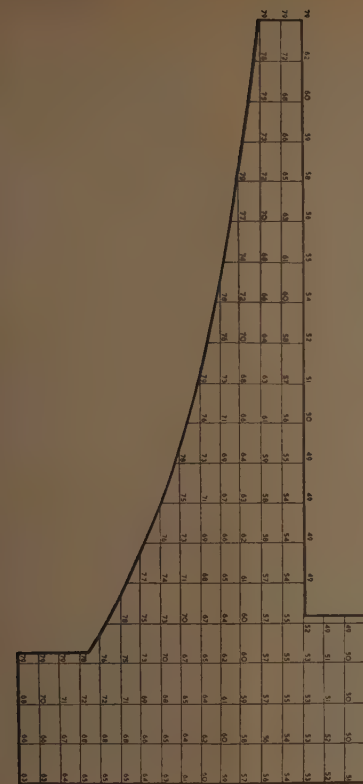


FIG. 30a.—TEMPERATURE. MEAN ANNUAL CONDITIONS, LESS 5°

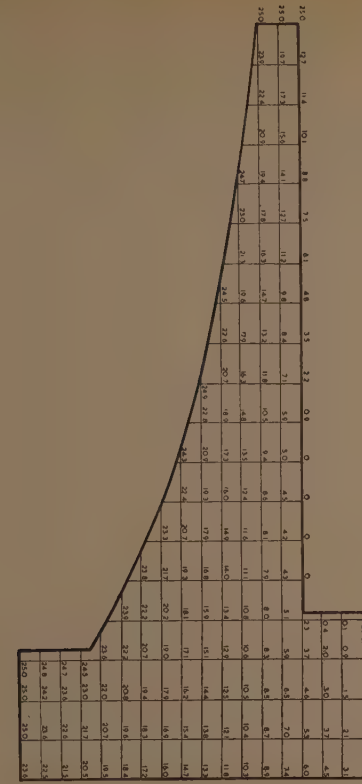


FIG. 30b.—TEMPERATURE. MAXIMUM DEVIATION FROM MEAN ANNUAL CONDITIONS

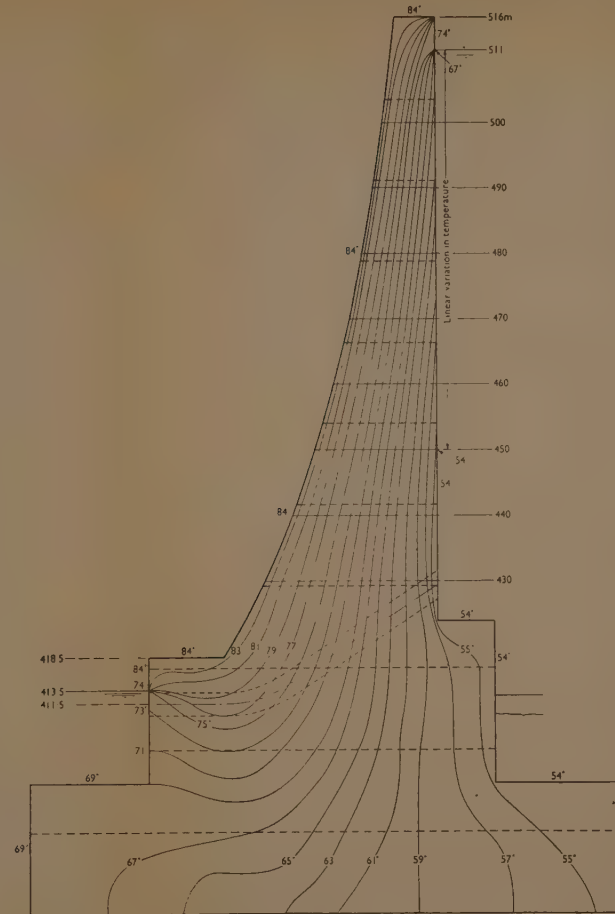


FIG. 30c.—TEMPERATURE CONTOURS. MEAN ANNUAL CONDITIONS

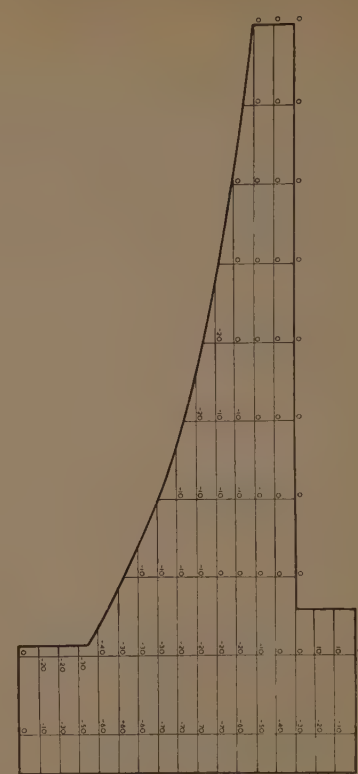


FIG. 31a.—THERMAL STRESSES. RADIAL STRESS r_r



FIG. 31b.—THERMAL STRESSES. TANGENTIAL STRESS t_t



FIG. 31c.—THERMAL STRESSES. VERTICAL STRESS z_z

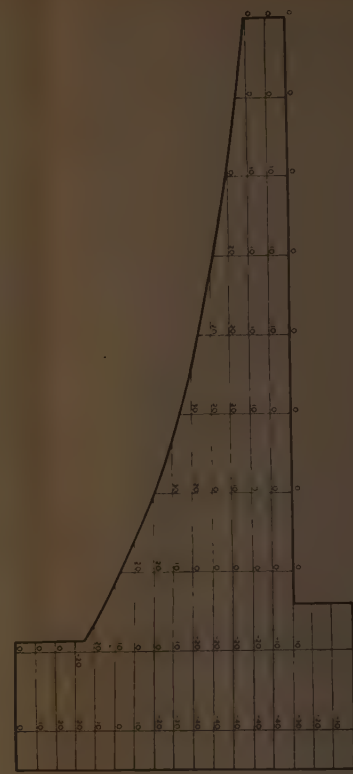


FIG. 31d.—THERMAL STRESSES. SHEAR STRESS z_r

The partial differential equations which govern the displacements are :

$$\left. \begin{aligned} - \sigma) & \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) + \frac{1 - 2\sigma}{2} \left(\frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{3 - 4\sigma}{2} \cdot \frac{1}{r^2} \cdot \frac{\partial w}{\partial \theta} \\ & + \frac{1}{2} \cdot \frac{\partial^2 v}{\partial r \partial z} + \frac{1}{2} \cdot \frac{1}{r} \cdot \frac{\partial^2 w}{\partial r \partial \theta} = 0, \\ - \sigma) & \frac{\partial^2 v}{\partial z^2} + \frac{1 - 2\sigma}{2} \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial v}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 v}{\partial \theta^2} \right) + \frac{1}{2} \cdot \frac{1}{r} \cdot \frac{\partial u}{\partial z} \\ & + \frac{1}{2} \cdot \frac{1}{r} \cdot \frac{\partial^2 w}{\partial \theta \partial z} + \frac{1}{2} \cdot \frac{\partial^2 u}{\partial r \partial z} = 0, \\ - \sigma) & \frac{1}{r^2} \cdot \frac{\partial^2 w}{\partial \theta^2} + \frac{1 - 2\sigma}{2} \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial w}{\partial r} - \frac{w}{r^2} + \frac{\partial^2 w}{\partial z^2} \right) + \frac{3 - 4\sigma}{2} \cdot \frac{1}{r^2} \cdot \frac{\partial u}{\partial \theta} \\ & + \frac{1}{2} \cdot \frac{1}{r} \cdot \frac{\partial^2 u}{\partial r \partial \theta} + \frac{1}{2} \cdot \frac{1}{r} \cdot \frac{\partial^2 v}{\partial \theta \partial z} = 0. \end{aligned} \right\} \quad (2)$$

}

These equations must be solved subject to certain boundary conditions as follows. Referring to Fig. 32, each part of the boundary is denoted by its intersection with the central section; thus EF is all the sloping curved part of the air face, and BC the horizontal water-face platform. The boundary conditions on each of the surfaces are then :

$$\left. \begin{aligned} \text{on AB and CD: } & \widehat{rr} = -p, \quad \widehat{zr} = 0, \quad \widehat{r\theta} = 0 \\ \text{on BC: } & \widehat{zz} = -p, \quad \widehat{zr} = 0, \quad \widehat{\theta z} = 0, \\ \text{on DE and FG: } & \widehat{zz} = 0, \quad \widehat{zr} = 0, \quad \widehat{\theta z} = 0, \\ \text{on GH: } & \widehat{rr} = 0, \quad \widehat{zr} = 0, \quad \widehat{r\theta} = 0, \\ \text{on EF: } & \widehat{rr} \cos \gamma + \widehat{zr} \sin \gamma = 0, \\ & \widehat{zz} \sin \gamma + \widehat{zr} \cos \gamma = 0, \\ & \widehat{r\theta} \cos \gamma + \widehat{\theta z} \sin \gamma = 0, \\ \text{on HA: } & u = 0, \quad v = 0, \quad w = 0, \end{aligned} \right\} \quad (3)$$

}

where γ is the inclination of the curved boundary to the z -direction. The conditions on HA are assumed to hold also in the foundation outside the limits of Fig. 32.

First-order finite differences

Fig. 38 shows a typical node 0 of the network with the six adjacent nodes 1, 2, 3, 4, 5, and 6 connected with it by mesh lines in the co-ordinate directions. The mesh-length in the r -direction is h , that in the z -direction is $2h$, and that in the θ -direction is $2h(r/\bar{r})$, where $\bar{r}$ is the radius at the water face, and r that at 0. Standard finite-difference expressions for the derivatives of any quantity q at 0 in terms of its values at adjacent nodes are:

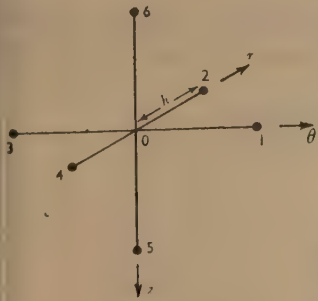


FIG. 38

$$\left. \begin{aligned} \left(\frac{\partial q}{\partial r} \right)_0 &= \frac{q_2 - q_1}{2h}, \\ \left(\frac{\partial q}{\partial z} \right)_0 &= \frac{q_6 - q_5}{4h}, \\ \text{and } \left(\frac{\partial q}{\partial \theta} \right)_0 &= \frac{r}{2h} \frac{q_1 - q_3}{4h} \end{aligned} \right\} \quad (4)$$

}

When the derivatives of u, v , and w in (1) are replaced by finite-difference approximations of type (4), the following formulae for the stress-components at a typical node 0 are obtained:

$$\left. \begin{aligned}
 (\widehat{rr})_0 &= \frac{E}{(1+\sigma)(1-2\sigma)} \left[(1-\sigma) \frac{u_2 - u_4}{2h} + \sigma \left(\frac{u_0}{r_0} + \frac{\bar{r}}{r_0} \cdot \frac{w_1 - w_3}{4h} + \frac{v_5 - v_6}{4h} \right) \right], \\
 (\widehat{\theta\theta})_0 &= \frac{E}{(1+\sigma)(1-2\sigma)} \left[(1-\sigma) \left(\frac{u_0}{r_0} + \frac{\bar{r}}{r_0} \cdot \frac{w_1 - w_3}{4h} \right) + \sigma \left(\frac{u_2 - u_4}{2h} + \frac{v_5 - v_6}{4h} \right) \right], \\
 (\widehat{zz})_0 &= \frac{E}{(1+\sigma)(1-2\sigma)} \left[(1-\sigma) \frac{v_5 - v_6}{4h} + \sigma \left(\frac{u_2 - u_4}{2h} + \frac{u_0}{r_0} + \frac{\bar{r}}{r_0} \cdot \frac{w_1 - w_3}{4h} \right) \right], \\
 (\widehat{\theta z})_0 &= \frac{E}{2(1+\sigma)} \left[\frac{\bar{r}}{r_0} \cdot \frac{v_1 - v_3}{4h} + \frac{w_5 - w_6}{4h} \right], \\
 (\widehat{zr})_0 &= \frac{E}{2(1+\sigma)} \left[\frac{u_5 - u_6}{4h} + \frac{v_3 - v_4}{2h} \right], \\
 (\widehat{r\theta})_0 &= \frac{E}{2(1+\sigma)} \left[\frac{w_3 - w_4}{2h} - \frac{w_0}{r_0} + \frac{\bar{r}}{r_0} \cdot \frac{u_1 - u_3}{4h} \right].
 \end{aligned} \right\} \quad (5)$$

TWO-DIMENSIONAL "TUMBLER" SOLUTION

The "tumbler" problem

This approximate two-dimensional solution assumes that the dam is part of a complete solid of revolution subject to axially symmetric loading. It is much easier to compute than the three-dimensional problem, since all quantities are independent of θ ; also the displacement w and the shear stress-components θz and $r\theta$ vanish throughout.

Equations (1) then simplify to:

$$\left. \begin{aligned}
 \frac{1+\sigma}{E} \widehat{rr} &= \frac{\partial u}{\partial r} + \frac{\sigma}{1-2\sigma} \left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} \right), \\
 \frac{1+\sigma}{E} \widehat{\theta\theta} &= \frac{u}{r} + \frac{\sigma}{1-2\sigma} \left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} \right), \\
 \frac{1+\sigma}{E} \widehat{zz} &= \frac{\partial v}{\partial z} + \frac{\sigma}{1-2\sigma} \left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} \right), \\
 \frac{1+\sigma}{E} \widehat{zr} &= \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right).
 \end{aligned} \right\} \quad (6)$$

and equations (2) to:

$$\left. \begin{aligned}
 (1-\sigma) \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) + \frac{1-2\sigma}{2} \cdot \frac{\partial^2 u}{\partial z^2} + \frac{1}{2} \cdot \frac{\partial^2 v}{\partial r \partial z} &= 0, \\
 (1-\sigma) \frac{\partial^2 v}{\partial z^2} + \frac{1-2\sigma}{2} \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial v}{\partial r} \right) + \frac{1}{2} \cdot \frac{\partial^2 u}{\partial r \partial z} + \frac{1}{2} \cdot \frac{1}{r} \cdot \frac{\partial u}{\partial z} &= 0.
 \end{aligned} \right\} \quad (7)$$

These must be solved subject to the first two of the three boundary conditions given in (3) on each of the various parts of the boundary.

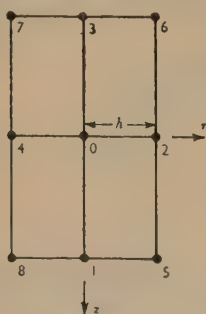


FIG. 39

Finite-differences

Fig. 39 shows any node 0 of the two-dimensional network and standard finite-difference expressions for the first and second derivatives of any quantity q at 0 in terms of nodal values of that quantity are:

$$\left. \begin{aligned}
 \left(\frac{\partial q}{\partial z} \right)_0 &= \frac{q_1 - q_3}{4h}, & \left(\frac{\partial q}{\partial r} \right)_0 &= \frac{q_2 - q_4}{2h}, \\
 \left(\frac{\partial^2 q}{\partial z^2} \right)_0 &= \frac{q_1 + q_3 - 2q_0}{4h^2}, \\
 \left(\frac{\partial^2 q}{\partial r^2} \right)_0 &= \frac{q_2 + q_4 - 2q_0}{h^2}, \\
 \left(\frac{\partial^2 q}{\partial z \partial r} \right)_0 &= \frac{q_5 - q_6 + q_7 - q_8}{8h^2}.
 \end{aligned} \right\} \quad (8)$$

Residuals

On replacing all derivatives of u and v in equations (7) by finite-difference expressions (8) the following equations result:

$$\left. \begin{aligned} F_{u_0} &\equiv \left(1 + \frac{h}{2r_0}\right)u_2 + \left(1 - \frac{h}{2r_0}\right)u_4 + \frac{1 - 2\sigma}{8(1 - \sigma)}(u_1 + u_3) \\ &\quad - \left(\frac{9 - 10\sigma}{4(1 - \sigma)} + \frac{h^2}{r_0^2}\right)u_0 + \frac{1}{16(1 - \sigma)}(v_5 - v_6 + v_7 - v_8) = 0, \\ \text{and} \\ F_{v_0} &\equiv \left(1 + \frac{h}{2r_0}\right)v_2 + \left(1 - \frac{h}{2r_0}\right)v_4 + \frac{1 - \sigma}{2(1 - 2\sigma)}(v_1 + v_3) - \frac{3 - 5\sigma}{1 - 2\sigma}v_0 \\ &\quad + \frac{1}{8(1 - 2\sigma)}(u_5 - u_6 + u_7 - u_8) + \frac{1}{4(1 - 2\sigma)} \cdot \frac{h}{r_0}(u_1 - u_3) = 0. \end{aligned} \right\} \quad \dots (9)$$

F_{u_0} and F_{v_0} are termed residuals and are defined by the expressions in (9). The aim of relaxation is to discover values of u and v at every node of the network such that the values of the two residuals at that node are both zero, as (9) shows they should be. Any values of u and v at the various nodes can be assumed and whatever guesses are made, corresponding values of F_u and F_v can be calculated from (9). If the guesses are correct the residuals will be zero; if incorrect, the non-zero values of the residuals are a measure of the error in the guess. Relaxation consists in reducing and ultimately liquidating the residuals corresponding with any initial guess.

Fictitious values

Fig. 40a represents conditions for nodes on AB or CD and Fig. 40b those for nodes on GH. Now node 2 in (a) and node 4 in (b) lie outside the boundary and are termed fictitious points; any values which may be attached to them are fictitious values. If the boundary conditions on AB or CD are expressed in terms of u and v by using equations (6) and then in finite-difference form by relations (8), they can be shown to yield appropriate values for fictitious displacements at node 2:

$$\left. \begin{aligned} u_2 &= u_4 - \frac{2\sigma}{1 - \sigma} \cdot \frac{h}{r_0}u_0 - \frac{\sigma}{2(1 - \sigma)}(v_1 - v_3) - \frac{2h(1 + \sigma)(1 - 2\sigma)}{1 - \sigma} \cdot \frac{p_0}{E}, \\ v_2 &= v_4 - \frac{1}{2}(u_1 - u_3), \end{aligned} \right\} \quad \dots (10)$$

while the boundary conditions imposed on GH yield values for fictitious displacements at node 4:

$$\left. \begin{aligned} u_4 &= u_2 + \frac{2\sigma}{1 - \sigma} \cdot \frac{h}{r_0}u_0 + \frac{\sigma}{2(1 - \sigma)}(v_1 - v_3), \\ v_4 &= v_2 + \frac{1}{2}(u_1 - u_3). \end{aligned} \right\} \quad \dots (11)$$

and

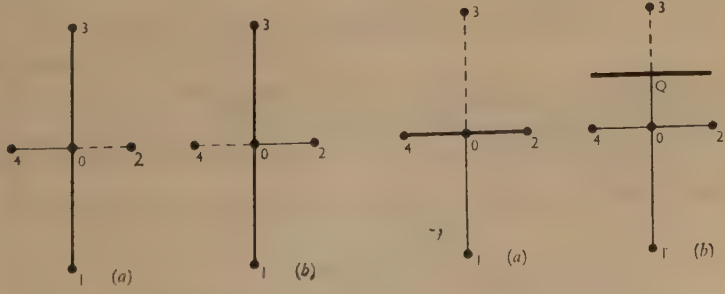


FIG. 40

FIG. 41

Similarly at a node 0 on DE in Fig. 41a the boundary conditions yield values for fictitious displacements at node 3:

$$\left. \begin{aligned} u_3 &= u_1 + 2(v_2 - v_4) \\ v_3 &= v_1 + \frac{4\sigma}{1-\sigma} \cdot \frac{h}{r_0} u_0 + \frac{2\sigma}{1-\sigma} (u_2 - u_4) \end{aligned} \right\} \dots \dots (12)$$

Fig. 41b represents conditions on boundaries BC and FG, which are *not* mesh lines of the net. Node 3 is fictitious and the mesh-line 03 intersects the boundary at Q where:

$$\begin{aligned} 0Q/03 &= \xi \\ \text{or} \quad 0Q &= \xi \cdot 2h. \end{aligned}$$

On BC the value of ξ is 0.1452 and on FG it is 0.2742. The second boundary condition (3) which holds at a point such as Q on BC or FG implies that:

$$\left(\frac{\partial u}{\partial z} \right)_Q + \left(\frac{\partial v}{\partial r} \right)_Q = 0. \dots \dots (13)$$

$$\text{By Taylor's expansion} \quad \left(\frac{\partial u}{\partial z} \right)_Q = \left(\frac{\partial u}{\partial z} \right)_0 - 2\xi h \left(\frac{\partial^2 u}{\partial z^2} \right)_0 + \dots \dots$$

for which the finite-difference approximation is:

$$\begin{aligned} \left(\frac{\partial u}{\partial z} \right)_Q &= \frac{u_1 - u_3}{4h} - 2\xi h \frac{u_1 + u_3 - 2u_0}{4h^2} \\ &= \frac{(1 - 2\xi)u_1 + 4\xi u_0 - (1 + 2\xi)u_3}{4h} \dots \dots (14) \end{aligned}$$

In the first instance $\left(\frac{\partial v}{\partial r} \right)_Q$ is replaced by $\left(\frac{\partial v}{\partial r} \right)_0$ to obtain the approximation:

$$\left(\frac{\partial v}{\partial r} \right)_Q = \frac{v_3 - v_4}{2h},$$

which leads to the formula:

$$u_3 = \frac{1 - 2\xi}{1 + 2\xi} u_1 + \frac{4\xi}{1 + 2\xi} u_0 + \frac{2}{1 + 2\xi} (v_2 - v_4), \dots \dots (15)$$

for the fictitious value u_3 . Similarly the first boundary condition (3) leads to:

$$\left. \begin{aligned} v_3 &= \frac{1 - 2\xi}{1 + 2\xi} v_1 + \frac{4\xi}{1 + 2\xi} v_0 + \frac{4\sigma}{1 - \sigma} \cdot \frac{h}{r_0} \cdot \frac{1}{1 + 2\xi} u_0 \\ &\quad + \frac{2\sigma}{1 - \sigma} \cdot \frac{1}{1 + 2\xi} (u_2 - u_4) + \frac{4h(1 + \sigma)(1 - 2\sigma)}{1 - \sigma} \cdot \frac{1}{1 + 2\xi} \cdot \frac{p_Q}{E}, \end{aligned} \right\} \dots (16)$$

for the fictitious value v_3 . On FG the pressure p is zero; on the water face, ABCD, it is fixed by the top water level at 5 m below the crest of the dam.

Fig. 42 represents conditions on the curved boundary EF. Point 4 is fictitious and the mesh-line 04 cuts the boundary at Q. The two boundary conditions are best re-arranged

so that each contains only one of the r -derivatives $\frac{\partial u}{\partial r}$ and $\frac{\partial v}{\partial r}$, which is achieved if they are

$$\left. \begin{aligned} \text{written} \quad & \widehat{rr} - \widehat{zz} \tan^2 \gamma = 0 \\ \text{and} \quad & (1 - \sigma \sec^2 \gamma) \widehat{zr} - \tan \gamma \{ \sigma \widehat{rr} - (1 - \sigma) \widehat{zz} \} = 0. \end{aligned} \right\}$$

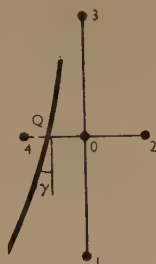


FIG. 42

in terms of u and v these become:

$$\left. \begin{aligned} -\sigma \sec^2 \gamma \frac{\partial u}{\partial r} + \sigma(1 - \tan^2 \gamma) \frac{u}{r} + (\sigma \sec^2 \gamma - \tan^2 \gamma) \frac{\partial v}{\partial z} &= 0, \\ \text{and } (1 - \sigma \sec^2 \gamma) \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \right) + 2 \tan \gamma \left(\sigma \frac{u}{r} + \frac{\partial v}{\partial z} \right) &= 0, \end{aligned} \right\} \dots \dots (17)$$

Conditions which hold at Q in Fig. 42.

In the first of (17) quantities $\left(\frac{\partial u}{\partial r} \right)_Q$ and u_Q are replaced by the finite-difference approximations to their Taylor expansions:

$$\left. \begin{aligned} \left(\frac{\partial u}{\partial r} \right)_Q &= \left(\frac{\partial u}{\partial r} \right)_0 - \eta h \left(\frac{\partial^2 u}{\partial r^2} \right)_0 + \dots \dots \dots \\ u_Q &= u_0 - \eta h \left(\frac{\partial u}{\partial r} \right)_0 + \frac{\eta^2 h^2}{2} \left(\frac{\partial^2 u}{\partial r^2} \right)_0 - \dots \dots \dots \\ \text{e., } \left(\frac{\partial u}{\partial r} \right)_Q &= \frac{(1 - 2\eta)u_2 + 4\eta u_0 - (1 + 2\eta)u_4}{2h}, \\ \text{and } u_Q &= (1 - \eta^2)u_0 - \frac{\eta(1 - \eta)}{2}u_2 + \frac{\eta(1 + \eta)}{2}u_4, \end{aligned} \right\} \dots \dots (18)$$

where $\eta = 0Q/04$. Similarly in the second of (17) the replacement is:

$$\left(\frac{\partial v}{\partial r} \right)_Q = \frac{(1 - 2\eta)v_2 + 4\eta v_0 - (1 + 2\eta)v_4}{2h}.$$

The other terms in both of (17) are replaced, in the first instance, by finite-difference approximations to their values at 0; thus $\left(\frac{\partial v}{\partial z} \right)_Q$ is replaced by $\frac{v_1 - v_3}{4h}$. Then fictitious values u_4 and v_4 can be calculated from:

$$\left. \begin{aligned} [1 - \sigma \sec^2 \gamma] \{ (1 - 2\eta)u_2 + 4\eta u_0 - (1 + 2\eta)u_4 \} \\ + \sigma(1 - \tan^2 \gamma) \frac{h}{r_0} \{ 2(1 - \eta^2)u_0 - \eta(1 - \eta)u_2 + \eta(1 + \eta)u_4 \} \\ + \frac{\sigma \sec^2 \gamma - \tan^2 \gamma}{2} (v_1 - v_3) &= 0 \\ \text{and } [1 - \sigma \sec^2 \gamma] \{ (1 - 2\eta)v_2 + 4\eta v_0 - (1 + 2\eta)v_4 \} \\ + \frac{1 - \sigma \sec^2 \gamma}{2} (u_1 - u_3) + 4\sigma \tan \gamma \frac{h}{r_0} u_0 + \tan \gamma (v_1 - v_3) &= 0. \end{aligned} \right\} \dots \dots (19)$$

In obtaining (15), (16), and (19), certain terms at points Q on the boundary have been replaced by finite-difference approximations at 0. This is convenient, and adequate in the first stages of relaxation. When the solution is nearing convergence, however, this approximation is easily refined. In Fig. 41b, for example, it was sufficient at first to assume a quantity q at Q to be q_0 . Eventually this is refined by taking

$$q_Q = (1 - \xi^2)q_0 - \frac{\xi(1 - \xi)}{2}q_1 + \frac{\xi(1 + \xi)}{2}q_3 \quad \dots \quad (20)$$

Similarly, in Fig. 42, the value of a quantity q at Q was first taken to be q_0 , but eventually,

$$q_Q = (1 - \eta^2)q_0 - \frac{\eta(1 - \eta)}{2}q_2 + \frac{\eta(1 + \eta)}{2}q_4 \quad \dots \quad (21)$$

these formulae are identical, and the latter has already been demonstrated in the second of (18).

Special treatment at corner points

Peculiar difficulties present themselves at corners of the section. Near B, illustrated in Fig. 43a, formulae already obtained enable fictitious values of the displacements to

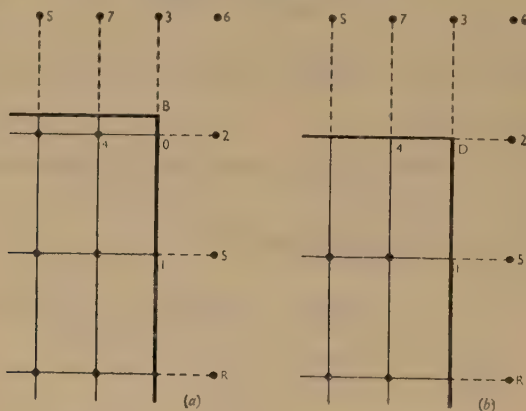


FIG. 43

be determined at all the fictitious nodes (2, 3, 5, 7, etc.) shown, except those at the point 6, which are also needed for evaluating the residuals F_v and F_u at 0. A number of methods have been investigated to overcome this difficulty; they all lead to substantially the same results and the one presented here has the merit of simplicity. By extrapolation from u -values already found, either at nodes 2, 5, and R, or at 3, 7, and S, two formulae:

$$\left. \begin{aligned} u_6 &= 3u_2 - 3u_5 + u_R \\ u_6 &= 3u_3 - 3u_7 + u_8 \end{aligned} \right\} \quad \dots \quad (22)$$

are obtained. It is not to be expected, certainly in the initial stages of the relaxation, that they will yield a common answer, but reasonable agreement may be expected, especially as the solution converges.

Both formulae are used throughout and the mean value of u_6 adopted; v_6 is determined from similar formulae.

Fig. 43b illustrates a different situation at the corner D which is itself a mesh-point. Here there is an additional difficulty, for fictitious values of v_2 and u_3 are not unique, since formulae (10) for v_2 and (12) for u_3 both demand that:

$$v_2 - \frac{1}{2}u_3 = v_4 - \frac{1}{2}u_1 \quad \dots \quad (23)$$

There is thus a degree of freedom in the choice of c_2 and u_3 and they can be chosen so as to minimize the discrepancies between the two pairs of values of u_6 and v_6 obtained from (22) and similar formulae.

The same type of difficulty at other corners can be dealt with similarly.

The relaxation process

The numerical process consists first in making an initial guess as to u and v at all nodes of the network which lie within or on the boundary of the dam. The various formulae are then used to calculate consistent values of u and v at the fictitious points just outside the boundary; these automatically ensure the satisfaction of the boundary conditions, and their existence enables residuals to be computed at all nodes in the section from formulae (9). If the guess is correct these residuals will all be zero; in practice this is unlikely, and the individual nodal values of u and v must be systematically adjusted until all the residuals have been liquidated and made negligibly small.

Relaxation patterns are therefore constructed to show how the residuals change value due to a unit increase in either u or v at a typical node. Fig. 44 contains two such patterns, (a) showing the changes made to the residuals F_u at 0 and the neighbouring nodes consequent on an increment $\Delta u = 1$ applied at 0, and (b) showing the changes made to the residuals F_v at 0 and the neighbouring nodes consequent on an increment $\Delta v = 1$ at 0. These patterns have been constructed with reference to the defining expressions (9).

It is best to alternate periods of u - and v -relaxation. For a start, residuals F_u should be liquidated by use of increments Δu and pattern (a) in Fig. 44, residuals F_v being temporarily ignored. When residuals F_u are sufficiently reduced, the fictitious values should all be recalculated and the residuals F_v evaluated. These in turn are liquidated by use of increments Δv and pattern (b) of Fig. 44, the residuals F_u being temporarily ignored. When the residuals F_v have been made small, the fictitious values must again be recalculated; then the residuals F_u should be recalculated and their liquidation continued. After a number of periods of alternate u - and v -relaxation the process converges to yield values of u and v with all residuals negligibly small.

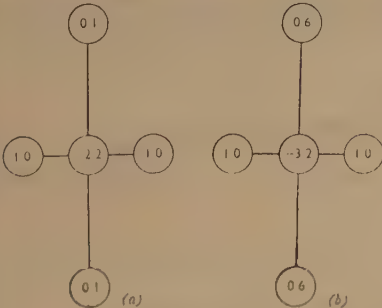


FIG. 44

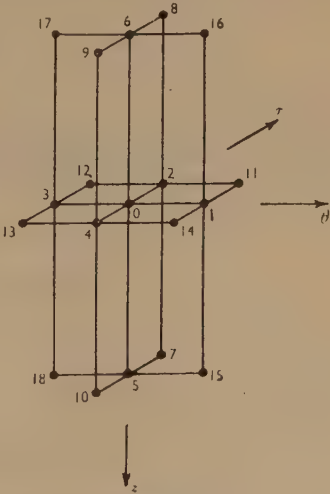


FIG. 45

THE THREE-DIMENSIONAL SOLUTION

The techniques developed in connexion with the tumbler solution can be extended to solve the three-dimensional problem. Fig. 45 represents a node 0 with the neighbouring nodes of the space lattice 1 to 18.

Residuals

Replacement of the governing equations (2) by their finite-difference approximations leads to three residual formulae at 0:

$$\left. \begin{aligned}
 F_{uo} &\equiv \left(1 + \frac{h}{2r_0}\right)u_2 + \left(1 - \frac{h}{2r_0}\right)u_4 + \frac{1-2\sigma}{8(1-\sigma)}(u_5 + u_6) \\
 &\quad + \frac{1-2\sigma}{8(1-\sigma)} \cdot \frac{\bar{r}^2}{r_0^2}(u_1 + u_3) - \left(\frac{9-10\sigma}{4(1-\sigma)} + \frac{1-2\sigma}{4(1-\sigma)} \cdot \frac{\bar{r}^2}{r_0^2} + \frac{h^2}{r_0^2}\right)u_0 \\
 &\quad - \frac{3-4\sigma}{8(1-\sigma)} \cdot \frac{h\bar{r}}{r_0^2}(w_1 - w_3) + \frac{1}{16(1-\sigma)}(v_7 - v_8 + v_9 - v_{10}) \\
 &\quad + \frac{1}{16(1-\sigma)} \cdot \frac{\bar{r}}{r_0}(w_{11} - w_{12} + w_{13} - w_{14}), \\
 F_{ro} &\equiv \left(1 + \frac{h}{2r_0}\right)v_2 + \left(1 - \frac{h}{2r_0}\right)v_4 + \frac{1-\sigma}{2(1-2\sigma)}(v_5 + v_6) \\
 &\quad + \frac{1}{4} \cdot \frac{\bar{r}^2}{r_0^2}(v_1 + v_3) - \left(\frac{3-5\sigma}{1-2\sigma} + \frac{1}{2} \cdot \frac{\bar{r}^2}{r_0^2}\right)v_0 + \frac{1}{4(1-2\sigma)} \cdot \frac{h}{r_0}(u_5 - u_6) \\
 &\quad + \frac{1}{16(1-2\sigma)} \cdot \frac{\bar{r}}{r_0}(w_{15} - w_{16} + w_{17} - w_{18}) + \frac{1}{8(1-2\sigma)}(u_7 - u_8 + u_9 \\
 &\quad \quad \quad - u_{10}), \\
 F_{wo} &\equiv \left(1 + \frac{h}{2r_0}\right)w_2 + \left(1 - \frac{h}{2r_0}\right)w_4 + \frac{1}{4}(w_5 + w_6) + \frac{1-\sigma}{2(1-2\sigma)} \cdot \frac{\bar{r}^2}{r_0^2}(w_1 + w_3) \\
 &\quad - \left(\frac{5}{2} + \frac{1-\sigma}{1-2\sigma} \cdot \frac{\bar{r}^2}{r_0^2} + \frac{h^2}{r_0^2}\right)w_0 + \frac{3-4\sigma}{4(1-2\sigma)} \cdot \frac{h\bar{r}}{r_0^2}(u_1 - u_3) \\
 &\quad - \frac{1}{8(1-2\sigma)} \cdot \frac{\bar{r}}{r_0}(u_{11} - u_{12} + u_{13} - u_{14}) \\
 &\quad \quad \quad + \frac{1}{16(1-2\sigma)} \cdot \frac{\bar{r}}{r_0}(v_{15} - v_{16} + v_{17} - v_{18}).
 \end{aligned} \right\} \quad (24)$$

Fictitious values

The complete boundary conditions (3) lead to formulae for displacements at all fictitious nodes, thus: On the vertical surface CD, which is a mesh-surface, node 2 is fictitious and:

$$\left. \begin{aligned}
 u_2 &= u_4 - \frac{2\sigma}{1-\sigma} \cdot \frac{h}{r_0}u_0 - \frac{\sigma}{2(1-\sigma)}(v_5 - v_6) \\
 &\quad - \frac{\sigma}{2(1-\sigma)}(w_1 - w_3) - \frac{2h(1+\sigma)(1-2\sigma)}{1-\sigma} \cdot \frac{p_0}{E}, \\
 v_2 &= v_4 - \frac{1}{2}(u_5 - u_6), \\
 w_2 &= w_4 + \frac{2h}{r_0}w_0 - \frac{1}{2}(u_1 - u_3).
 \end{aligned} \right\} \quad (25)$$

On the vertical surface GH, also a mesh-surface, node 4 is fictitious and:

$$\left. \begin{aligned}
 u_4 &= u_2 + \frac{2\sigma}{1-\sigma} \cdot \frac{h}{r_0}u_0 + \frac{\sigma}{2(1-\sigma)}(v_5 - v_6) + \frac{\sigma}{2(1-\sigma)} \cdot \frac{\bar{r}}{r_0}(w_1 - w_3), \\
 v_4 &= v_2 + \frac{1}{2}(u_5 - u_6), \\
 w_4 &= w_2 - \frac{2h}{r_0}w_0 + \frac{1}{2} \cdot \frac{\bar{r}}{r_0}(u_1 - u_3).
 \end{aligned} \right\} \quad (26)$$

On the vertical surface AB, which intersects 02 at Q where $0Q = \frac{1}{2}h$, node 2 is fictitious and :

$$\left. \begin{aligned} \left(1 + \frac{3\sigma}{8(1-\sigma)} \cdot \frac{h}{r_Q}\right) u_2 &= \left(1 - \frac{3\sigma}{4(1-\sigma)} \cdot \frac{h}{r_Q}\right) u_0 + \frac{\sigma}{8(1-\sigma)} \cdot \frac{h}{r_Q} u_4 \\ &\quad - \frac{\sigma}{4(1-\sigma)} (v_5 - v_6) - \frac{\sigma}{4(1-\sigma)} \cdot \frac{\bar{r}}{r_0} (w_1 - w_3) \\ &\quad - \frac{h(1-2\sigma)(1+\sigma)}{1-\sigma} \cdot \frac{p_Q}{E}, \\ v_2 &= v_0 - \frac{1}{4}(u_5 - u_6), \end{aligned} \right\} \quad (27)$$

$$\left(1 - \frac{3}{8} \cdot \frac{h}{r_Q}\right) w_2 = \left(1 + \frac{3}{4} \cdot \frac{h}{r_Q}\right) w_0 - \frac{1}{8} \cdot \frac{h}{r_Q} w_4 - \frac{1}{4} \cdot \frac{\bar{r}}{r_0} (u_1 - u_3).$$

On the crest DE, which is a mesh-surface, node 6 is fictitious and :

$$\left. \begin{aligned} u_6 &= u_5 + 2(v_2 - v_1), \\ v_6 &= v_5 + \frac{4\sigma}{1-\sigma} \cdot \frac{h}{r_0} u_0 + \frac{2\sigma}{1-\sigma} (u_2 - u_4) + \frac{\sigma}{1-\sigma} \cdot \frac{\bar{r}}{r_0} (w_1 - w_3), \\ w_6 &= w_5 + \frac{\bar{r}}{r_0} (v_1 - v_3). \end{aligned} \right\} \quad (28)$$

On the horizontal surface BC and FG, neither of which is a mesh-plane, but which intersect the line 06 at Q where $0Q = \xi \cdot 2h$, node 6 is fictitious and :

$$\left. \begin{aligned} u_6 &= \frac{1-2\xi}{1+2\xi} u_5 + \frac{4\xi}{1+2\xi} u_0 + \frac{2}{1+2\xi} (v_2 - v_4), \\ v_6 &= \frac{1-2\xi}{1+2\xi} v_5 + \frac{4\xi}{1+2\xi} v_0 + \frac{4\sigma}{1-\sigma} \cdot \frac{h}{r_0} \cdot \frac{1}{1+2\xi} u_0 + \frac{2\sigma}{1-\sigma} \cdot \frac{1}{1+2\xi} (u_2 - u_4) \\ &\quad + \frac{\sigma}{1-\sigma} \cdot \frac{\bar{r}}{r_0} \cdot \frac{1}{1+2\xi} (w_1 - w_3) + \frac{4h(1+\sigma)(1-2\sigma)}{1-\sigma} \cdot \frac{1}{1+2\xi} \cdot \frac{p_Q}{E}, \\ w_6 &= \frac{1-2\xi}{1+2\xi} w_5 + \frac{4\xi}{1+2\xi} w_0 + \frac{\bar{r}}{r_0} \cdot \frac{1}{1+2\xi} (v_1 - v_3). \end{aligned} \right\} \quad (29)$$

For plane BC, $\xi = 0.5726$ and for plane FG, $\xi = 0.1371$.

On the curved surface EF, which intersects 04 at Q where $0Q = \eta h$, node 4 is fictitious and u_4 , v_4 , and w_4 are found from :

$$\left. \begin{aligned} (1 - \sigma \sec^2 \gamma) \{ (1 - 2\eta) u_2 + 4\eta u_0 - (1 + 2\eta) u_4 \} \\ + \sigma (1 - \tan^2 \gamma) \frac{h}{r_Q} \{ 2(1 - \eta^2) u_0 - \eta(1 - \eta) u_2 + \eta(1 + \eta) u_4 \} \\ + \frac{(\sigma \sec^2 \gamma - \tan^2 \gamma)}{2} (v_5 - v_6) + \frac{\sigma(1 - \tan^2 \gamma)}{2} \cdot \frac{\bar{r}}{r_0} (w_1 - w_3) = 0, \\ (1 - \sigma \sec^2 \gamma) \{ (1 - 2\eta) v_2 + 4\eta v_0 - (1 + 2\eta) v_4 \} + \frac{1 - \sigma \sec^2 \gamma}{2} (u_5 - u_6) \\ + 4\sigma \tan \gamma \frac{h}{r_0} u_0 + \tan \gamma (v_5 - v_6) + \sigma \tan \gamma \frac{\bar{r}}{r_0} (w_1 - w_3) = 0, \\ (1 - 2\eta) w_2 + 4\eta w_0 - (1 + 2\eta) w_4 - \frac{h}{r_Q} \{ 2(1 - \eta^2) w_0 - \eta(1 - \eta) w_2 + \eta(1 + \eta) w_4 \} \\ + \frac{\bar{r}}{2r_0} (u_1 - u_3) + \frac{\tan \gamma}{2} \cdot \frac{\bar{r}}{r_0} (v_1 - v_3) + \frac{\tan \gamma}{2} (w_5 - w_6) = 0. \end{aligned} \right\} \quad (30)$$

The special treatment at corner points follows exactly that described for the tumbler solution.

Relaxation process

This is a direct extension of that already described. Periods of u -, v -, and w -relaxation are alternated, using the relaxation patterns shown in Figs 46a to c. Pattern (a) shows the changes in residuals F_u consequent upon an increment $\Delta u = 1$ at node 0, while (b) and (c) show the corresponding patterns for liquidating the v - and w -residuals respectively.

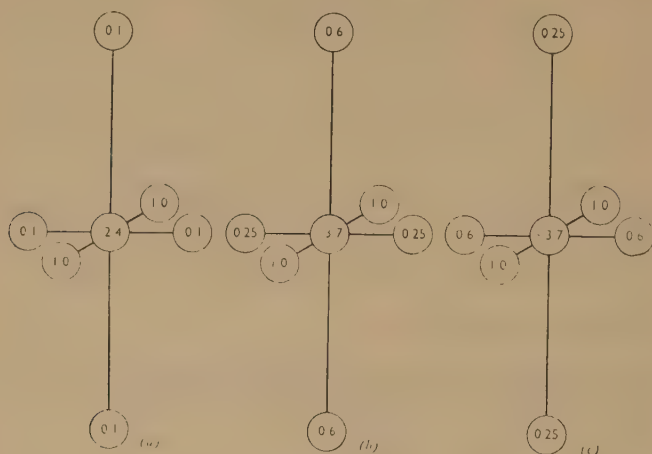


FIG. 46

After each period of relaxation the fictitious values should be recalculated and also the set of residuals due for liquidation in the next period of relaxation.

GRAVITY STRESSES

The gravity stresses in the separate monoliths can be obtained by relaxation of the two-dimensional equation either in terms of displacements, or of an Airy function χ .¹⁰ The latter treatment showed that the stresses $\hat{r}r$ and $\hat{r}z$ nowhere exceeded 30 lb. sq. in. and that the usual engineer's method was quite adequate.

TEMPERATURE DISTRIBUTION AND DEVIATIONS

Figs 29a and b show the mean surface temperatures and the maximum deviations from them. The data did not warrant more than a two-dimensional study of the problems involved.

Steady-state temperature

The distribution through the dam is obtained by solving the partial differential equation :

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial z^2} = 0 \quad . \quad . \quad . \quad (31)$$

subject to the given temperature conditions on the boundary. In finite-difference form equation (31) becomes :

$$F_0 \equiv \left(1 + \frac{h}{2r_0}\right)t_2 + \left(1 - \frac{h}{2r_0}\right)t_4 + \frac{1}{4}(t_1 + t_3) - \frac{5}{2}t_0 = 0. \quad . \quad . \quad (32)$$

Special formulae at nodes near the boundary are obtained in the usual way.⁷

The thermal stress problem

The thermal stresses in an axially-symmetric solid can be expressed* in terms of two stress functions, ϕ and ψ , which satisfy a pair of simultaneous second-order partial differential equations:

$$\left. \begin{aligned} \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \cdot \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} &= 0, \\ \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \cdot \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} &= \frac{\partial^2 \phi}{\partial z^2} + \frac{E\alpha}{1-\sigma} r \frac{\partial t}{\partial r}, \end{aligned} \right\} \quad \dots (33)$$

where α is the temperature coefficient of expansion of the material, taken to be 0.0000046. In terms of ϕ and ψ the four components of stress at any point are:

$$\left. \begin{aligned} \widehat{rr} &= \frac{1}{r} \cdot \frac{\partial}{\partial r}(\phi + \psi) - \frac{1}{r^2}(\psi + (1-\sigma)\phi) - \frac{E\alpha t}{1-\sigma}, \\ \widehat{\theta\theta} &= \frac{\sigma}{r} \cdot \frac{\partial \phi}{\partial r} + \frac{1}{r^2}(\psi + (1-\sigma)\phi) - \frac{E\alpha t}{1-\sigma}, \\ \widehat{zz} &= -\frac{1}{r} \cdot \frac{\partial \psi}{\partial r}, \\ \widehat{zr} &= \frac{1}{r} \cdot \frac{\partial \psi}{\partial z}, \end{aligned} \right\} \quad \dots (34)$$

and the component displacements u and v are related with ϕ and ψ by the equations:

$$\left. \begin{aligned} u &= \frac{1+\sigma}{E} \cdot \frac{1}{r}(\psi + (1-\sigma)\phi), \\ \frac{\partial v}{\partial z} &= -\frac{1+\sigma}{E} \cdot \frac{1}{r} \cdot \frac{\partial}{\partial r}(\psi + \sigma\phi) + \frac{1+\sigma}{1-\sigma} \alpha t, \\ \frac{\partial v}{\partial r} &= \frac{1+\sigma}{E} \cdot \frac{1}{r} \cdot \frac{\partial}{\partial z}(\psi - (1-\sigma)\phi). \end{aligned} \right\} \quad \dots (35)$$

The numerical solution of equations (33) was performed on a relaxation net of sides $h = 3.1$ m in the r -direction and $4h = 12.4$ m in the z -direction, as seen in Figs 31a to d, Plate 2. At a typical node 0 residuals are defined by the finite-difference approximations to equations (33):

$$\left. \begin{aligned} F_{\phi_0} &\equiv \left(1 - \frac{h}{2r_0}\right)\phi_2 + \left(1 + \frac{h}{2r_0}\right)\phi_4 + \frac{1}{16}(\phi_1 + \phi_3) - \frac{17}{8}\phi_0 = 0, \\ F_{\psi_0} &\equiv \left(1 - \frac{h}{2r_0}\right)\psi_2 + \left(1 + \frac{h}{2r_0}\right)\psi_4 + \frac{1}{16}(\psi_1 + \psi_3) - \frac{17}{8}\psi_0 \\ &\quad + \frac{1}{16}(2\phi_0 - \phi_1 - \phi_3) - \frac{E\alpha}{1-\sigma} \cdot \frac{hr_0}{2}(t_1 - t_3) = 0. \end{aligned} \right\} \quad \dots (36)$$

The stresses are obtained from ϕ and ψ by means of the finite-difference approximations to (34):

$$\left. \begin{aligned} (\widehat{rr})_0 &= \frac{1}{2hr_0}(\phi_2 + \psi_2 - \phi_4 - \psi_4) - \frac{1}{r_0^2}(\psi_0 + (1-\sigma)\phi_0) - \frac{E\alpha}{1-\sigma}t_0, \\ (\widehat{\theta\theta})_0 &= \frac{\sigma}{2hr_0}(\phi_2 - \phi_4) + \frac{1}{r_0^2}(\psi_0 + (1-\sigma)\phi_0) - \frac{E\alpha}{1-\sigma}t_0, \\ (\widehat{zz})_0 &= -\frac{1}{2hr_0}(\psi_2 - \psi_4), \\ (\widehat{zr})_0 &= \frac{1}{8hr_0}(\psi_1 - \psi_3). \end{aligned} \right\} \quad \dots (37)$$

* See reference 27 on p. 230.

In the expressions (36) and (37) appropriate values of t are taken from the solution recorded in Fig. 30b.

The thermal stress boundary conditions

To compute residuals from expressions (36) at nodes which are either on or near the boundary, the boundary conditions determine formulae for fictitious values of ϕ and ψ at nodes outside the boundary. In the thermal stress problem the part of the boundary ABCDEFGH (Fig. 32) is unloaded and the stress boundary conditions are:



FIG. 47

$$\left. \begin{aligned} \widehat{zz} \sin(v, r) + \widehat{zr} \cos(v, r) &= 0 \\ \text{and } \widehat{rr} \cos(v, r) + \widehat{rz} \sin(v, r) &= 0 \end{aligned} \right\} \quad (38)$$

where v (Fig. 47) denotes the outward drawn normal and (v, r) the angle, measured anti-clockwise, from the v -direction to the r -direction. If distance s along the boundary is measured from A toward H, on substitution from (34), the first of (38) reduces to $\partial\psi/\partial s = 0$, so that ψ is a constant on ABCDEFGH, which it is convenient to take as zero. Thus on this part of the boundary ψ is known and special formulae near the boundary are found in the usual way.⁷

The second of (38) in terms of ϕ and ψ becomes:

$$\left(\frac{\partial\phi}{\partial r} + \frac{\partial\psi}{\partial r} - \frac{1-\sigma}{r}\phi - \frac{E\alpha}{1-\sigma}rt \right) \cos(v, r) + \frac{\partial\psi}{\partial z} \sin(v, r) = 0. \quad (39)$$

Making use of the first of (38), equation (39) can be written in the alternative form:

$$\left(\frac{\partial\phi}{\partial r} + \frac{\partial\psi}{\partial r} - \frac{1-\sigma}{r}\phi - \frac{E\alpha}{1-\sigma}rt \right) \cos^2(v, r) + \frac{\partial\psi}{\partial r} \sin^2(v, r) = 0. \quad (40)$$

Either condition (39) or (40) is used on the various parts of these boundaries as follows:

(i) On the vertical boundaries AB, CD and GH, which are all mesh-lines, and at nodes lying on them $\psi = 0$, therefore no residual F_ψ need be expressed. To obtain a residual F_ϕ the first and second of (33) are added, then since $\psi = 0$, and hence $\partial^2\psi/\partial z^2 = 0$, on AB, CD, and GH:

$$\frac{\partial^2(\phi + \psi)}{\partial r^2} - \frac{1}{r} \cdot \frac{\partial(\phi + \psi)}{\partial r} - \frac{E\alpha}{1-\sigma} r \frac{\partial t}{\partial r} = 0.$$

The finite-difference approximation to this equation leads to the residual formula for nodes on AB, CD, and GH:

$$F_{\phi_0} \equiv \left(1 - \frac{h}{2r_0}\right)(\phi_2 + \psi_2) + \left(1 + \frac{h}{2r_0}\right)(\phi_4 + \psi_4) - 2\phi_0 - \frac{E\alpha}{1-\sigma} \cdot \frac{hr_0}{2}(t_3 - t_1) = 0, \quad (41)$$

to be used instead of the first of (36). (41) contains the fictitious value $\phi_2 + \psi_2$ for nodes on AB and CD, and $\phi_4 + \psi_4$ for nodes on GH. Formulae for these are derived from the boundary condition (39), and since on both these boundaries $\sin(v, r) = 0$:

$$(\phi_2 + \psi_2) - (\phi_4 + \psi_4) = 2(1-\sigma)\frac{h}{r_0}\phi_0 + \frac{2E\alpha}{1-\sigma}hr_0t_0. \quad (42)$$

This may be used to determine $\phi_2 + \psi_2$ at nodes on AB and CD, and $\phi_4 + \psi_4$ at nodes on GH.

(ii) On the crest DE, also a mesh-line, again only a residual F_ϕ is required. Since

$\psi = 0$ it follows also that $\partial\psi/\partial r = \partial^2\psi/\partial r^2 = 0$ and hence, on adding the first and second of (33) :

$$\frac{\partial^2\phi}{\partial r^2} - \frac{1}{r} \cdot \frac{\partial\phi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} - \frac{E\alpha}{1-\sigma} r \frac{\partial t}{\partial r} = 0,$$

which leads to the residual formula at nodes on DE :

$$F_{\phi_0} \equiv \left(1 - \frac{h}{2r_0}\right)\phi_2 + \left(1 + \frac{h}{2r_0}\right)\phi_4 - 2\phi_0 + \frac{1}{16}(\psi_1 + \psi_3) - \frac{E\alpha}{1-\sigma} \cdot \frac{hr_0}{2}(t_2 - t_4) = 0, \quad \dots \quad (43)$$

instead of the first of (36). This formula contains the fictitious value ψ_3 . But the boundary condition (39) reduces to $\partial\psi/\partial z = 0$ since $\cos(\nu, r) = 0$, and therefore $\psi_3 = \psi_1$.

(iii) On the horizontal boundaries BC and FG, which are not mesh-lines, the node 3 is fictitious, cf. Fig. 41b,* when 0 lies inside the boundary and O3 intersects the boundary at a point Q where $OQ = \xi.03$. On BC the value of ξ is 0.5726 whilst on FG it is 0.1371. A residual formula F_{ϕ} is found in the same way as the formulae (41) and (43):

$$F_{\phi_0} \equiv \left(1 - \frac{h}{2r_0}\right)(\phi_2 + \psi_2) + \left(1 + \frac{h}{2r_0}\right)(\phi_4 + \psi_4) + \frac{1}{16}(\psi_1 + \psi_3) - 2\phi_0 - \frac{17}{8}\psi_0 - \frac{E\alpha}{1-\sigma} \cdot \frac{hr_0}{2}(t_2 - t_4) = 0. \quad \dots \quad (44)$$

In this formula ψ_3 is fictitious but the boundary condition (39) reduces at Q to $\partial\psi/\partial z = 0$ and since also $\psi = 0$ at Q, the formulae :

$$\psi_3 = \left(\frac{1-\xi}{1+\xi}\right)^2 \psi_1 \quad \text{and} \quad \psi_0 = \left(\frac{\xi}{1+\xi}\right)^2 \psi_1 \quad \dots \quad (45)$$

are obtained for the fictitious value ψ_3 and the selvage value ψ_0 .

(iv) On the curved boundary EF, the node 4 is fictitious, cf. Fig. 42,† and a formula for the fictitious value ϕ_4 is required. The alternative form (40) for the boundary condition can be rewritten

$$\frac{\partial\phi}{\partial r} - \frac{1-\sigma}{r}\phi + \frac{\partial\psi}{\partial r} \sec^2(\nu, r) - \frac{E\alpha}{1-\sigma} r t = 0$$

and its finite-difference approximation which holds at Q leads to the formula :

$$\begin{aligned} &(1 + 2\eta)\phi_4 - (1 - 2\eta)\phi_2 - 4\eta\phi_0 \\ &+ (1 - \sigma)\frac{h}{r_Q}\{\eta(1 + \eta)\phi_4 - \eta(1 - \eta)\phi_2 + 2(1 - \eta^2)\phi_0\} \\ &+ 2 \sec^2(\nu, r)\left(\frac{\eta}{1 + \eta}\psi_2 - \frac{1 + \eta}{\eta}\psi_0\right) + \frac{2E\alpha}{1 - \sigma} hr_Q t_Q = 0 \quad \dots \quad (46) \end{aligned}$$

for the fictitious ϕ_4 , where $\eta = OQ/O4$

(v) On the base HA, $u = v = 0$; hence $\partial v/\partial r = 0$ and so from the first and third of (35) :

$$\left. \begin{aligned} \phi &= -\frac{\psi}{1-\sigma} \\ \frac{\partial\psi}{\partial z} - (1-\sigma)\frac{\partial\phi}{\partial z} &= 0. \end{aligned} \right\} \quad \dots \quad (47)$$

and

* The situation here is similar to that illustrated in Fig. 41b, but there the mesh-length in the z -direction was $2h$, whereas here it is $4h$.
† The mesh-length in the z -direction in Fig. 42 is $2h$; here it is $4h$.

The first of these expresses ϕ -values directly as multiples of the ψ -values; and no residuals $F\phi$ are required. To define a residual $F\psi$ the two equations in (33) are combined in the form:

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \cdot \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = (1 - \sigma) \frac{\partial^2 \phi}{\partial z^2} - \sigma \left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \cdot \frac{\partial \phi}{\partial r} \right) + \frac{E\alpha}{1 - \sigma} r \frac{\partial t}{\partial r},$$

and so at a node 0 on HA:

$$F\psi_0 \equiv \left(1 - \frac{h}{2r_0}\right)(\psi_2 + \sigma\phi_2) + \left(1 + \frac{h}{2r_0}\right)(\psi_4 + \sigma\phi_4) - \frac{17}{8}\psi_0 \\ + \frac{1}{16}(\psi_1 + \psi_3 - (1 - \sigma)(\phi_1 + \phi_3)) - \frac{17\sigma - 1}{8}\phi_0 - \frac{E\alpha}{1 - \sigma} \cdot \frac{hr_0}{2}(t_2 - t_4) = 0. \quad (48)$$

This formula contains the fictitious value $\psi_1 - (1 - \sigma)\phi_1$, for which the second of the boundary conditions (47) provides the formula:

$$\psi_1 - (1 - \sigma)\phi_1 = \psi_3 - (1 - \sigma)\phi_3. \quad . \quad . \quad . \quad . \quad (49)$$

The Paper, which was received on 6 August, 1955, is accompanied by eleven photographs and fifty sheets of diagrams, from some of which the half-tone page plates, folding Plates 1, 2, 3, and 4, and the Figures in the text have been prepared, and by an Appendix.

Discussion

Mr G. M. Binnie (Partner, Messrs Binnie, Deacon, and Gourley, Consulting Engineers, London) opened the discussion by making a few remarks about the design of the Dokan dam in general and the Authors' work in connexion with it. When designing an arch dam, he said, one of the first decisions to be made was whether it was going to serve purely as a structure for impounding water or whether it was desired to have outlets such as hydro-electric penstocks incorporated in it. If it was decided that the dam should be used only for impounding, then the cupola type of design¹¹ was attractive; but if, as at Dokan, it was necessary to incorporate several large penstocks in the dam, then a vertical upstream face had advantages, or at any rate they had thought so, and that was the explanation of the vertical upstream face.

The general siting of the abutments, the use of a vertical face with a radius of 120 m (394 ft) on the upstream side of the arch, and the dimensions of the base block had all been dictated by practical considerations such as the geology and topography of the site. Fig. 30c, Plate 2, showed the manner in which the penstocks, five in number, had been incorporated in the base block. This had facilitated trashrack design, also gate design, installation, operation, and maintenance. Having gone as far as it had been possible to go on practical considerations, they had been left with the problem of deciding how thick the dam ought to be at various levels to withstand successfully water, temperature, and earthquake loadings, and so on, with the desired factor of safety and within whatever might be considered to be the permissible limiting stresses. That was a much more complex matter. It was possible to make certain arbitrary assumptions, such as treating the dam as a series of inclined arches, in order to simplify the calculations, or to undertake a three-dimensional analysis strictly in accordance with the elastic theory, as had been described in the Paper.

Concrete, of course, had construction joints, temperature stresses during hydration of cement, swelling in contact with water, and so on, and was subject to shrinkage and

creep, and therefore it could not be said that it was homogeneous, isotropic, and monolithic in accordance with the requirements of the elastic theory. The same also applied to the rock in the foundations. He had met some engineers who claimed that a three-dimensional analysis was not really worth the labour spent on it, because of those difficulties, but he thought that that was a counsel of despair, perpetuating for all time design on arbitrary and empirical methods.

Whatever might be said against the application of the elastic theory to arch dams, it provided the only rational basis on which it was possible to compare mathematically the relative merits of one design against another. They had therefore decided that, if it was at all possible, a three-dimensional analysis in accordance with the elastic theory should be undertaken for the Dokan dam. Such an unusual problem, however, was quite outside their normal office activities and presented very great difficulties; but, with the consent and enthusiastic support of their clients, they had been fortunate enough to interest Professor Pippard and his colleagues in their problems, and the result of their work was given in the Paper.

One explanation which should perhaps be given to assist in an understanding of the Paper was that where the Authors had referred to stresses at grouting temperature it should be borne in mind that cooling pipes were being embedded in the dam, so that with the aid of refrigerated water the dam could be cooled down to any desired temperature before the vertical joints between the monoliths were grouted up under pressure.

The judicious use of both experimental and mathematical methods of investigation was an outstanding feature of the Authors' work, and the Paper contained a wealth of material. It was difficult to pick out any one subject as more outstanding than another, but probably one of the most interesting and even unique features was that, for the centre sections in the river, the theoretical stresses had been given in accordance with the elastic theory in the interior of the dam, instead of on the usual basis of linear distribution from face to face. This was perhaps the first time that consulting engineers had ever been presented with such a gift. It was also most interesting that in the model work for the design D.3, the final design, an exact replica of the dam and of the rock round the foundations had been made—exact apart from the omission of a few "frills" which did not matter; functionally it was an exact model. The only difference which really existed between the model and the prototype from a theoretical point of view was the difference in Poisson's ratio, which for rubber was 0.5 and for concrete was anybody's guess; perhaps 0.15 would be a fair figure, whilst for rock the figure might be about the same. The clue which the Authors had given towards the end of the Paper to this elusive problem of Poisson's ratio was one of the most exhilarating things which had been produced for a long time.

The warmest thanks of his firm were due to the Authors for their assistance in dealing with his firm's particular problems, but the Authors' research work had a much more important and far wider bearing than on the Dokan dam. It was also an important contribution to the theory of complex structures.

Sir Richard Southwell had been intensely interested by the Paper, which brought back, for him, memories of 45 years ago; for it was the famous "Drapers' Company Memoir" of Atcherley and Karl Pearson¹² which first made him realize the existence of a thing called research—that the theorems taught to him had not been delivered, once for all, on tables of stone. Controversy had followed of which he knew nothing at the time, though in later years he had heard something from J. S. Wilson of his work with models.¹³ They had given little support to the Atcherley-Pearson theory, as Professor Pippard had said a few years ago.¹⁴

Ever since, Sir Richard had hoped that the theoretical side would be developed far enough to permit a real comparison with Wilson's models. L. F. Richardson had gone a long way in 1910,¹⁵ but not far enough to provide a conclusive test: so when, in 1940, Dr L. Fox and he had found the biharmonic equation tractable,¹⁶ and hence the straight two-dimensional dam, he had hoped that someone would make the application. In 1947

Dr Zienkiewicz had done so,¹⁷ and in view of the agreement between experiment and his computations it became apparent that Karl Pearson would have to "take the count." (This, too, had been suggested by Professor Pippard.)

But the problem under discussion now involved three variables (for the shape was not of the Authors' choosing but dictated by Nature, by rock-formation) and, as such, was immensely harder. Being at the moment engaged in the completion of a book reviewing the progress of relaxation methods since 1940 (when his first account of them was published), Sir Richard had been really delighted to learn of the appearance of what might be called its coping-stone. It had appeared too late for detailed notice of it in the book, but he would still be able to add a note of reference to it; which delighted him because he thought its problem an ideal example with which to illustrate the value to engineers (whose shapes were inexorably dictated, always) of methods that yielded numerical results at once, instead of formulae that still required to be laboriously translated into numbers. (The advantage possessed by algebraic solutions, of including "parameters" which might be given *any* values, was reduced when one's interest centred in a particular dam, not in all dams of a "family.") No one could dispute, he thought, that in the present instance relaxation methods had "delivered the goods." That made him very happy, and he would like to express his thanks for the invitation to this Meeting.

Professor A. D. Ross (Professor of Civil Engineering, King's College, University of London) commented on the choice of the material for the models and the question of Poisson's ratio. It seemed that for work of the kind in question it was possible to use a material such as plaster, which could be cast and which had approximately the correct value of Poisson's ratio, but which, because of its relatively high elastic modulus, required

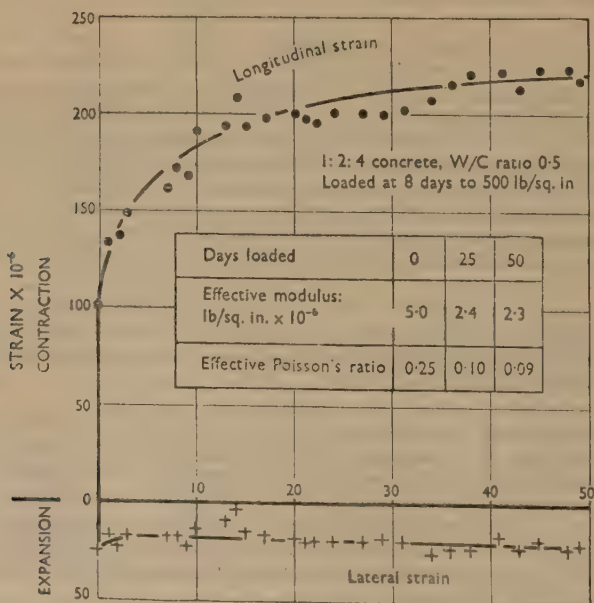


FIG. 48

to be loaded by mercury or some mechanical device in order to obtain displacements which could be measured without too much difficulty. That technique, of course, had been used in a number of model studies in the past. Rubber had the great advantage of a low elastic modulus, so that water loading was practicable; but its Poisson's ratio was

0.5, which made its behaviour somewhat unusual. A more emphatic way of putting it was to say that the ratio of bulk modulus to shear modulus, for structural materials such as steel and concrete, was about 2, whereas for rubber it was between 3,000 and 4,000. That underlined the difference in the behaviour of those two types of material.

He had, therefore, on his first reading of the Paper, been critical of Fig. 17, in which radial displacements at the crest of the full-size dam had been deduced from the model, apparently ignoring the large difference in Poisson's ratio for rubber and concrete. He had realized later, however, that the Authors had not omitted to consider the matter and the discussion on pp. 226-229 he regarded as of immense importance. Although the Authors disclaimed any rigorous proof, the general argument was convincing. If a large variation of Poisson's ratio, from 0.5 to 0.15, involved a change in radial and tangential displacements of only about 17%, it was clear that Poisson's ratio was not an influential factor, except for vertical displacements. It was possible, however, that it might have a more significant effect on stresses as distinct from displacements.

The question of Poisson's ratio was, he believed, of particular importance because of the behaviour of concrete when it was loaded for a long period. Fig. 48 showed a typical result for an experiment which had been performed many times. A sustained stress of 500 lb/sq. in. was applied to a concrete specimen. On loading, there was an immediate strain in the direction of the stress corresponding to a modulus of 5.0×10^6 lb/sq. in., and an immediate transverse strain corresponding to a Poisson's ratio of 0.25, which was rather a high figure. As the load was sustained, the longitudinal strain increased due to creep, while the lateral strain remained almost unchanged. If, therefore, the effective Poisson's ratio was defined as the ratio of the total lateral strain to the total longitudinal strain, it would be seen that it had diminished after 50 days under load to 0.09.

The behaviour under two-dimensional stressing was similar, as was shown in Table 14. Those specimens had been subjected to two equal compressive principal stresses and both the effective modulus and the effective Poisson's ratio had diminished. In view of the large reduction in effective Poisson's ratio in 50 or 60 days, it seemed clear that after a very long period it would have a probably negligible value. In all the results shown, drying shrinkage had been subtracted from the strains. In a dam, shrinkage would alter

TABLE 14.—VARIATION IN EFFECTIVE MODULUS (ALLOWING FOR CREEP) AND EFFECTIVE POISSON'S RATIO

Specimens stressed two-dimensionally.
(Shrinkage deducted.)

(a) 1 : 2 : 4 concrete. W/C ratio 0.6
Loaded at 21 days.

Loaded period—days	0	20	40	60
Effective modulus lb/sq. in. $\times 10^6$.	4.2	2.7	2.2	2.0
Effective Poisson's ratio	0.18	0.11	0.09	0.08

(b) 1 : 2 : 4 concrete. W/C ratio 0.5
Loaded at 8 days.

Loaded period—days	0	20	40	60
Effective modulus lb/sq. in. $\times 10^6$.	4.2	2.8	2.6	2.3
Effective Poisson's ratio	0.18	0.12	0.11	0.10

profoundly the stresses in the outer skin, but would have little effect on the major part of the interior.

It would seem, therefore, that concrete should be regarded as a material which had, on loading, a Poisson's ratio of about one-fifth, but when considering total strains under sustained loads the effective ratio might decline ultimately to a very small value. Since no single value for the ratio could be correct for all times, he would like to suggest that the error in assuming that Poisson's ratio was zero might be no greater than in assuming that it was 0.15. He realized, of course, that three-dimensional tests, under conditions comparable to those in the interior of a dam, would be necessary before such a suggestion could be justified, but he would be interested to have the Authors' comments on that. Presumably it would simplify the analysis.

Considering the calculation of stresses, it was clear that the actual value assumed for E , Young's modulus, did not really matter except in the case of temperature stresses. If too high a value of E was used, the displacements would be under-estimated, but that was automatically corrected in the computation of the stresses by the use of the same value of E . Whether it was taken as 5,000,000 or 1,000,000 lb/sq. in. therefore, did not seem to matter very much. That underlined the fact that, although creep altered the displacements, it did not alter the stresses directly; the dam merely functioned as an elastic structure with a reduced value of E .

Professor Ross thought, however, that the influence of Poisson's ratio was less obvious. The Authors had shown that radial and tangential displacements were not seriously affected by a variation of Poisson's ratio, but equation (1) in the Appendix suggested that an over-estimate of Poisson's ratio might result in the under-estimation of some of the stresses, and that, of course, must be guarded against.

Finally, it would be of great interest to hear a little more about the relaxation process. As he understood it, the displacements of the model had been used to estimate the initial configuration or starting-point for the relaxation. How good was that estimate? Were the residuals still so large as to merit the use of the special steps to speed the process? He felt sure that with no fewer than 350 stations disposed three-dimensionally in the half-network, any device of art or arithmetic which would reduce the number of relaxations required would be welcome.

He sincerely hoped that the Authors would be able to take their work further into the design stage. It would be interesting to know how the results of their analysis compared, for example, with the trial-load method. If they found it possible to formulate a workable method of design, it would mean that the very excellent piece of research which they had carried out would become generally available and applicable to dams other than Dokan.

Mr H. D. Morgan (Partner, Sir William Halcrow and Partners, Consulting Engineers) remarked that the mathematical treatment was analogous to that devised by Sir Isaac Newton for arriving at the roots of an equation by iteration, making successively closer approximations. The work checked itself for that reason, successive steps producing more and more accuracy, if that was desired.

He recollected another occasion when he had had cause to admire the mathematical dexterity of Miss Chitty, in connexion with a problem concerning guyed steel masts,¹⁸ where an expression of hideous complexity had been reduced to a series which converged so rapidly that only the first term was significant. That sort of thing naturally appealed strongly to a practising engineer.

He had a particular interest in the design of arch dams, and also a personal interest in the Dokan dam, for he had been a member of an international panel which had visited Iraq and the site in November 1952, in order to investigate and report on the proposals of Messrs Binnie, Deacon, and Gourley, his colleagues being Mr André Coyne, of France, and Mr S. C. Harper, of the United States, who had been Chief Engineer of the Bureau of Reclamation. Mr Harper had naturally been a stout supporter of the trial-load method of analysis as practised by the Bureau of Reclamation, and that had led to a certain amount of debate between them because Mr Morgan had not been and was not a great admirer

of that method of analysis when carried to the lengths advocated by the Bureau of Reclamation. He was not disposed to admit that it was the only approach to the subject.

On p. 202 the Authors had stated that "The most elaborate and detailed method of analysis so far developed is the trial-load method devised by the U.S. Bureau of Reclamation after many years of systematic research." He wished to say with all possible emphasis that that method had *not* been devised by the U.S. Bureau of Reclamation. The pioneer of that method of treatment had been Dr A. Stucky, who had been at the time assistant to Professor Gruner, and who published a Paper¹⁹ on the method of analysis into arch and cantilever elements. He had adopted the method of dividing the dam into a number of cantilever and arch elements and dividing the loads between them. At that time Mr Morgan had thought that the only other Paper of any value had been the one by William Cain²⁰ in 1921. The method had been used early in 1930 by Mr Morgan's firm in order to design an arch dam, the principal designer being the late Mr R. S. Cole. Mr Morgan had been Mr Cole's assistant and remembered very well how heavy the work involved had been, because the final result of the balancing process had been six simultaneous equations. They had taken six levels for the six arch elements, and had also considered a centre cantilever and two others. Merely for one solution there had been six simultaneous equations; the coefficients of p_1 had been of the order of thousands and of p_6 just a single digit, so that any attempt to eliminate by brute force was accompanied by astronomical numbers; it had been very difficult to cut off figures and approximate. They had spent about 2 weeks on the first trial and the result had not checked, so a search had had to be made for the mistake. However, after unearthing an old Brunsviga calculating machine it had become possible to solve the equations resulting from later trial calculations in about a day; but the work had been very heavy. Any attempt to use determinants only resulted in the same heavy work. It had all originated in that way, and the Americans had elaborated it to an extraordinary degree.

The Authors had stated the assumptions made in the trial-load method, and later they had referred to "The artificial device of dividing the dam into interpenetrating cantilevers and arches and then determining the stresses from resultant actions by formulae of doubtful validity." The validity was doubtful, but it was a fact that the accuracy of the results obtained by any method of analysis, including that advocated by the Authors, could not possibly be greater than that of the data on which they were based. That remark applied equally to the relaxation method.

The practice of the U.S. Bureau of Reclamation of giving results carried to four and even seven significant figures was in Mr Morgan's opinion very much to be deprecated. It was merely accuracy of figures. It would be fortunate if accuracy to two significant figures were obtained, and indeed that should satisfy any practising engineer, since it meant that his maximum error would be of the order of 10%. For a problem of the nature in question, and dealing with so many physical constants which were only approximately known, such as the effective elastic modulus of the concrete and rock, Poisson's ratio, and the like, the engineer would be lucky if his calculated results were reliable within an error of 10%. In a structure such as a bridge, where the engineer was dealing with a material of known and uniform properties, he could expect to be more accurate if he knew what his loads were, but then generally he did not know his loading with accuracy.

On p. 203 the Authors had remarked that experience in the application of the trial-load method was lacking in the United Kingdom. Mr Morgan did not feel inclined to accept that statement, because he thought that he had said enough to show that the method had been used in Britain, and in fact it had been used comparatively recently to design an arch dam which had been built in Portugal. In that case a simplified method had been used because, owing to the known inaccuracy of the data, it had been decided that it was useless to employ too much refinement, so an attempt had been made to simplify the method in such a way that the preliminary trials could be done rapidly, and only the final balancing required greater accuracy.

He was glad to see that the Authors were much more realistic than the Americans, and that their results in Table 2 were given, quite properly, only to three significant figures.

The agreement shown in Fig. 16 was remarkable and clearly indicated that the method of analysis was fundamentally correct. That, of course, related to the rubber model, for which the physical constants of the material were accurately known and for which the material was of constant quality throughout. In the case of a dam which was constructed by adding what were comparatively small increments of concrete, one by one, each one differing slightly from the rest and bonded more or less to the rest, it was nonsensical to expect to get ideal conditions and the same accuracy of prediction could not be expected. Nevertheless he thought that the Authors' method gave the right framework and enabled the engineer to know the order of what he was doing. There was some comfort from the fact that, at any rate so far as Mr Morgan's knowledge went, there had never yet been a failure of an arch dam.

Dr Charles Jaeger (English Electric Co. Ltd) emphasized the importance of the research which the Authors had carried out on the behaviour of large arch dams.

The basis for the first investigation for the Dokan dam had been an interpolation of data from two existing dams, the Rossens dam, which was 87 m (285 ft) high in a U-shaped valley, with a ratio of dam-length to height of 3.8, and the Tignes dam, 180 m (591 ft) high, built in a V-shaped valley with a ratio of only 1.6. In both cases the angle at the centre at crest level was 111° . The Rossens dam had been designed by H. Gicot and Max Ritter and the Tignes dam by André Coyne. It was very remarkable that when the main cross-sections of those two dams were reduced to the same height and superimposed, they nearly coincided. It had been felt that the conditions at Dokan permitted basing the first preliminary design for the dam there on those two dams, suitably adapted to the local conditions. The choice of the cross-section and central angle of the Dokan dam had therefore been influenced by examples from existing structures.

The thickness/height ratio in the preliminary project had been 0.31. The Dokan dam was a very bold design, but not exceptionally so; there were several Italian dams where the thickness/height ratio was 10–12%. They were all built in V-shaped valleys, which might account for their exceptional boldness. The Mauvoisin dam, 235 m (771 ft) high, and the Lienne dam, 160 m (525 ft) high, had ratios of 23% and 15.6% respectively. By far the most audacious dam which had been built was the Gage dam, 38 m (125 ft) high, designed by André Coyne, with a thickness of only 2.5 m ($8\frac{1}{2}$ ft), giving a ratio of only 6.7%. The ratio of crest length to dam height was 3.7, a figure which added further to the audacity of the design.

The result of the subsequent thorough investigations by the Authors had shown the basic assumptions for the Dokan dam to be reasonably sound.

There were many possible approaches to the arch dam problem, and the Authors had mentioned the two main lines of thought—the use of approximate methods, developed by Stucky, Bourgin, and Ritter,* and the more correct but very tedious American trial-load method. The approximate method of Ritter was a simplified trial-load analysis which would yield the essential result in a relatively short time. One calculator could go through the whole calculation in a few months, whereas the trial-load method required the work of a full team over a far longer period.

One approach to the arch-dam problem was to start with an approximate stress and strain analysis, as in the method of Ritter or Stucky, or a simplified trial-load analysis, the results of which were then compared with measurements on models. Sufficient data were available on the methods of calculation, the model technique, the materials to be used for construction, and the comparison of calculated and measured strain and stresses to make that approach a safe method. It would be found that the approximate methods neglected the shear stresses, but the magnitude of the principal stresses were nevertheless accurate, only their directions being incorrectly presented. In fact the direction of the principal stresses was inclined towards the abutments, but the approximate method would show them to be horizontal.

* A comprehensive bibliography is given in reference 2, which appears on p. 230.

An excellent example of that method of calculation was the Lienne dam by Gicot and Lardi, an account of which had been submitted to the Fifth Congress on Large Dams in Paris in June 1955. The Lienne thin arch dam was a difficult unsymmetrical structure. The model tests* had shown the influence of shear stresses and torsion on a dam of that type and the effect of shear inclining the lines of principal stresses towards the lateral abutments. But even in that difficult case the results of an approximate analysis had been satisfactory and reliable.

The great advantage of the use of model tests in any method of calculation was that it permitted a quick check of alternative designs.

The approach described by the Authors had gone a step further. They had considered the dam as having a thick shell and had solved the general equation of elasticity for that difficult case by relaxation methods. Recently scientists in several countries had attempted to solve that fundamental problem, and Italian and Swiss research work on the subject might be mentioned. The Authors had solved the problem with remarkable elegance, and the comparison with model tests had gone a long way towards elucidating the behaviour of the dam structure and determining the strain and stress patterns in arch dams. For many years experts in dam design had been of the opinion that such a calculation by relaxation methods should be attempted.

The value of the Paper was increased if it was compared with other modern publications on arch dam calculation and arch dam testing. The main drawback of such calculation was that it required the help of a highly specialized team of mathematicians, and could no longer be carried out by the engineers normally available in a consultant's office. Most of the work would be done by specialists who might have no intimate knowledge of the design of a dam as a whole or of the construction techniques and materials to be used. For many years approximate methods of calculation combined with model tests would receive the attention of the specialist in the case of works of lesser importance than the Dokan dam.

A stage had now been reached where the combined efforts of engineers in many countries would make it possible to estimate the stresses in a dam, to predict the reliability of any method of calculation, and to compare the results with those of model tests. A point had even been reached where model test results could be corrected to come near the truth, as shown in the Paper, where the effect on the strains of an excessively high Poisson's ratio was corrected to get the true value of the actual strains. Whatever the method of approach, approximate or exact, engineers now knew more or less where they stood and could use additional model tests to help them out. However, the final step to real knowledge of arch dam behaviour was still far away.

At the beginning of his remarks Dr Jaeger had said that the first investigation for the Dokan dam had been based on the interpolation of data from the Rossens and Tignes dams. There should be no better method than direct comparison with existing dams, all dams being considered as possible prototypes for new ones and yielding the necessary information for improved design. However, that was not entirely the case. Knowledge of the behaviour of real dams was very meagre compared with knowledge of the behaviour of dam models. Dozens of results from models were available for comparison and scrutiny, but publications dealing in detail with the results of strain and stress measurements on real dams were scarce. American, Portuguese, Swiss, and Italian publications showed the intricacy of the problems when deflexion and strains due to temperature variation, rock plasticity, and concrete shrinkage interfered with the elastic displacements and strains caused by variation of the hydrostatic load. The present Paper was a step forward of considerable importance, as was shown by the basic similarity of the results yielded by highly detailed mathematical analysis to those obtained by model tests. The next step, which was still to come, was a more general view on the behaviour of existing dams, compared with the results given by model tests.

* A new plastic material was used for the models of the Lienne dam.

Dr O. C. Zienkiewicz (Lecturer, Department of Engineering, University of Edinburgh) felt that the work presented by the Authors went a long way towards giving a more exact and logical method of approach than had hitherto been possible. He expected that the labour involved in the full relaxation solution greatly exceeded that of his own work on gravity dams completed some years ago under the guidance of two of the Authors.

The problem of arch dams was in one other respect more difficult than that of gravity dams. The wide range of profiles and valley sections made it practically impossible to generalize the results of stress distributions obtained for one case. The great usefulness of the present work lay in providing a standard against which any approximate methods of analysis might be checked.

He expressed satisfaction that the stresses due to water load and gravity and the thermal stresses had been completely separated, thus making it easier to follow the individual effects and adjust the treatment to suit the case. When he had first heard of the "tumbler" solution proposed for the analysis of the dam, he had been very sceptical and had not thought that the stresses would be comparable with those in a true arch dam, where the restraint of abutments was possibly greater than that of the foundation. It was very interesting to see, however, that the "tumbler" solution agreed reasonably well with the full solution and both showed the tendency for tensile stresses to develop on the downstream face about mid-height of the dam when the water load was applied. That tendency had been noted on previous models, including some in the United States in 1935, where the same kind of distribution had been shown.

Dr Zienkiewicz had tried to work out a comparison between the stresses obtained by "tumbler" analysis, by the full analysis, and also by the very approximate analysis sometimes used in preliminary designs of arch dams:—the cylinder method. It seemed that on grid 6, about mid-height of the dam, the cylinder stresses worked out at 475 lb/sq. in., the "tumbler" giving a variation from 400 to 375 lb/sq. in. for the tangential stress, and the full analysis 480 to 320 lb/sq. in. In that case the order of magnitude was roughly comparable by all methods, but on grid 12 the cylinder analysis gave a figure of the order of 520 lb/sq. in. for the tangential stress, whereas the "tumbler" analysis gave 190 to 230 lb/sq. in. and the full analysis 280 to 80 lb/sq. in. Both the "tumbler" and the full analysis gave answers which were of the same order and also very much smaller than those obtained by the cylinder method, because at the lower levels on dams the transition to cantilever stresses naturally occurred.

Referring to gravity stresses, it was not clear whether the analysis had been obtained by the "tumbler" method or if it had been done on a free-standing slice. The Authors followed the correct line in not applying the analysis to the dam as a monolithic structure, because individual blocks were separately built and there was no connexion between them up to the final grouting stage. The "tumbler" analysis, however, would give some tangential stresses, whereas the radial-slice analysis would not. The difference between the two approaches might not result in a measurable error.

It was stated in the Paper that at the beginning of the investigation it had been thought necessary to measure the displacements due to gravity in the model. He would have expected that they would have been entirely different from those found by the analysis, because of the effect of abutments, which was not introduced in nature.

Regarding the thermal stresses he did not think that the use of the "tumbler" analysis was altogether applicable. Professor Pippard had said that the "tumbler" analysis had not been used to give the thermal stresses near the abutments, but in Dr Zienkiewicz's opinion even in the centre sections the agreement would not be very good. He wondered what results were obtained by the *encastré* arch theory (which had been used for the calculation of the thermal abutment stresses) for the stresses at the centre section.

It might be an indiscreet question to ask, but he would like to know what direction the simplified theories were going to take in the future. He had been thinking about this problem for some time, and he wondered whether, in view of the fact that most of the stresses did not depart largely from a linear distribution on sections of the dam, the

application of elastic thin-shell theories would make the analysis easier and give at all comparable results with the full analysis. The case of the "tumbler" which corresponded, in the thin-shell theory, to the cylindrical tank built-in at the base, had been solved theoretically for a shell of uniform thickness and for one varying uniformly in section. A solution in which there was a variation of thickness in the same ratio as that at the Dokan Dam was being computed and it should be interesting to compare the stresses with those calculated by the Authors. It can be stated *a priori* that both distributions of stress would be very similar. The reversal of the bending moment and the tendency for tension to develop on the downstream face were indicated in the existing shell theory solutions. Had it been possible to measure strains on the model and to work out the stresses from them?

Mr Binnie had raised a point about the shape of the dam and had mentioned that the Home was a structure which was possibly economically preferable. Dr Zienkiewicz was not sure why the question of penstocks should have been a deciding factor. He would like to refer to the Enchanet arch dam recently constructed in France which was of extremely slender section and which included penstocks. That dam leaned pronouncedly in the downstream direction, so forming an effective dome and possibly resulting in a large economy of material.

Mr S. P. Christodoulides (Ove Arup and Partners) observed that the results obtained from the analysis were not compared with full-scale measurements on the concrete structure, and it could not be claimed that an opportunity was given to check the accuracy of the experimental and mathematical methods suggested by the Paper or

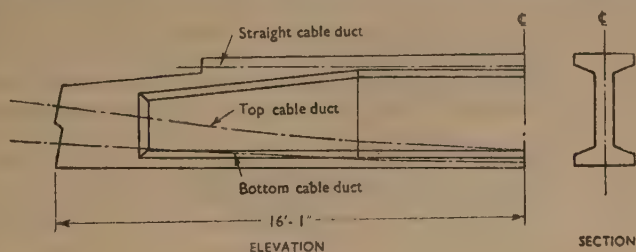


FIG. 49.—GANTRY BEAM

Above: details of full-scale beam
Below: fringe pattern for model

the quantitative value of the results obtained. If it was assumed that the accuracy of the methods suggested was comparable with that normally expected in the analysis of structures of similar engineering nature, the length of time taken (3 years) should be considered in comparison with other methods available.

A similar problem had been discussed in a previous Paper²¹. The analysis of the Errochty dam had been investigated by two-dimensional photoelastic methods, and it had been suggested²² that three-dimensional photoelastic work could have preferably been carried out by using the "Frozen Stress" technique.

Large true-to-shape scale models could now be made from synthetic resins, and "Araldite B" hot-setting resin had been extensively used in the last 3 years.²³ (See Fig. 49).

Araldite castings could be made in the laboratory using reasonable care and a thermostatically controlled oven, and the models could be machined fairly easily from them.

Araldite, like many other materials, was optically sensitive to stress. When plane polarized light (light vibrating in one plane) was transmitted through a stressed model made of such material, it split into two waves vibrating in the principal directions and propagating with different velocities. The "phase difference" between the two waves at the end of their travel could be measured and was proportional to the principal stress difference. By photoelastic methods both the directions and the algebraic difference of the principal stresses could be obtained at any point in any plane. By using those quantities, the separate values of the principal stresses could be derived.

Three-dimensional work could be carried out by using the "frozen stress" technique.²⁴ An attractive method of investigation suitable for structures like the Dokan dam would be the "shear slope" method.²² Such investigation could be completed within 2-3 months, and would involve the work of an investigator and his technician. That, of course, compared well with the 3 years taken by the Authors to produce the present investigation.

The accuracy of the "shear slope" method was high, and in the case of the gantry beam given in Fig. 49 it was consistent within 10%. Comparison with full-scale tests gave an error of 15%.²⁵

The elastic constants of Araldite as used in the "frozen stress" technique were:—

Young's modulus = 1,950 lb/sq. in.

Poisson's ratio $\approx$ 0.49 lb/sq. in.

Tensile strength = 280 lb/sq. in.

Its optical sensitivity was high (1.30 lb/sq. in./fringe/in. of thickness). Recent research carried out at the University College, London, on the effects of Poisson's ratio on three-dimensional systems, had shown that that was not serious.²⁶ The stress at any point did not vary by more than 10% of the maximum stress if Poisson's ratio was varied from 0.3 to 0.49. That variation was usually of the same order as the error involved in experimental work. It would, in that respect, be interesting to know to what accuracy the Authors' measurements on the rubber models had been.

Professor Allen, in reply, said that whilst he spoke with some trepidation as a mathematician in the home of civil engineering, he had no hesitation about the first remark he wished to make, which was to express his gratitude to civil engineers for the opportunity to co-operate on the research described in the Paper. He would like specifically to mention Mr Binnie, whose problem it was, and Professor Pippard, who had been first approached about the work which had been done at Imperial College. It had been a very stimulating experience, a challenge to relaxation technique, which, despite occasional setbacks, Professor Allen thought had been fairly met. Much more research remained to be tackled in the theory of the design of arch dams, but he thought the work described in the Paper indicated what should be a fruitful approach for further investigations.

One question, raised by Professor Ross, was whether there would be any simplification in relaxation work by taking Poisson's ratio to be zero. That would not, in fact, par-

ticularly simplify the calculations. The numerical data were fed directly into the relaxation method, and the particular values of those data could, by and large, be whatever the engineer chose to specify. The work was entirely arithmetical, and it did not much matter, so far as the series of calculations which had to be made was concerned, what the numbers were. There was, however, one exception to that statement and that was for the particular case where σ (Poisson's ratio) took the value $\frac{1}{2}$. That case was quite different from all others, because the factor $(\frac{1}{2} - \sigma)$ then vanished. The factor entered the mathematical equations in several places and their character changed completely when it vanished. Some research students at Imperial College were at present working on the analysis of a dam of a rather simplified section to find a solution when Poisson's ratio was $\frac{1}{2}$. Professor Ross had suggested that the analysis would be simplified in those circumstances; that was perhaps true of the formal *analysis*, but the numerical *solution* would appear to be more difficult.

Professor Ross had also asked whether an overestimation of Poisson's ratio would lead to an underestimation of the stresses. Professor Allen could not give a definite answer; to make certain he would like to work the problem through again with a different value of Poisson's ratio and see what resulted. He saw no evidence one way or the other in the basic equations, and his opinion was that a change of Poisson's ratio from 0.15 to 0.10, for example, would not materially alter the conclusions which the Authors had drawn in their results.

The Authors had been asked to say more about the relaxation application and the value of model experiments, and coupled with that there was the question, put by Dr Zienkiewicz, about how much time the actual relaxation calculations had taken. That was difficult to answer, and the reason for the difficulty might be appreciated if Professor Allen gave a brief history of the relaxation calculations. When the Authors had started the investigation, the idea of a full three-dimensional solution was not considered to be practically feasible; they had not previously tackled the determination of such a stress system in three dimensions. They had started on the "tumbler" solution, and their first thought had been to get a solution for the two-dimensional section before deciding on the next step. That work had been started before the model measurements had been available, so it had not been possible to take advantage of them at the beginning. The Authors had hoped to find stresses according to the "tumbler" method for various sections of the actual dam and then to be able later on to marry them up one with another; but they had soon realized that that was going to be an unprofitable course to pursue, and had decided to look seriously into the question of a possible three-dimensional solution.

When they came to tackle it they found that it was not so difficult as they had feared it might be. Nevertheless, the way in which they had, in fact, tackled the problem had been to find the "tumbler" solution first of all, and that had taken some time. They had worked with variables which were not the displacements but certain mathematical stress-functions, which it was unnecessary to discuss at the moment, and it was only later that they had discarded stress-functions and used the actual displacements in the dam. When they had turned to seek the full three-dimensional solution it had been the first attempt that they had made at a three-dimensional relaxation solution of the equations of elasticity. The Authors had not known how those equations would behave. They had tried various exploratory attacks; some had failed, and some had succeeded without being very promising. They had spent a great deal of time investigating the best way of applying relaxation to that type of problem before the final process was accepted.

That was a brief history of the relaxation work. The "tumbler" solution had been found for three different sections; there were two preliminary designs and the second had been slightly modified, leading to the final shape of the section. The time taken, once they had started three-dimensional work on the final design by their accepted process, was between 6 and 9 months. The nature of the work was such that only one individual, for the most part, could work on the project at any time. There had been occasions when a double shift was worked, but it was not really profitable to have more than one person in charge of the arithmetic at any one time.

Dr Zienkiewicz had referred to the thickness/height ratio of Dokan dam and other

dams. Professor Allen's view was that satisfactory dams could be designed with much thinner sections than that of the Dokan dam, but that would require the section to be such that the water face was not vertical. That the water face should be vertical had been a prerequisite in the Dokan design, and that, he thought, was the reason why the ratio was about 0.3. The advantage of the section suggested by Dr Zienkiewicz was that a component of the water pressure on the water face tended to produce a compressive stress in the vertical direction. Ideally, if the problem concerned only a dam to hold water, such a dam would have considerable curvature in both horizontal and vertical planes and then it should be possible to have a much smaller thickness/height ratio.

Mention had been made of the possibility of measuring strains on the model in order to help in the numerical calculations and in the subsequent calculation of stresses. In order to get the stresses from the strains it was necessary to have the strains in three directions. Presumably, measurements on a model could be made only in the tangential directions at the surface.

The Authors had been asked about the value of the model measurements as a preliminary to the numerical calculations. In relation to that, it would be true to say that on another occasion the model displacements would be extremely helpful to the mathematician, giving him his first guess at the answer. He would then have to correct that guess to satisfy the mathematical equations. The Authors' work, however, had been done in the first instance simultaneously with the model measurements, which had not been available to help them at the very beginning. They had been very glad of them later as a check to the "tumbler" solution; in the three-dimensional case they had had the model measurements, but by then they had also had the "tumbler" solutions. The great advantage of the model measurements then had been that they showed that the variation at radial displacement round the arch approximated closely to a sine-wave. To begin the three-dimensional problem, the "tumbler" values had been taken on the central section, and along each arch other values had been chosen according to a (complete) sine-wave distribution.

Professor Allen regretted that it had not been made clear that the gravity stresses were obtained for a free-standing slice, and not for a "tumbler" problem.

Professor Allen could in no way accept Mr Morgan's argument that any method of doubtful validity ever became justified solely by virtue of ignorance of precise data.

Miss Chitty, in answer to Mr Christodoulides's question about the order of accuracy of the measurements taken on the model, said that they had all been made by cathetometers carried on independent staging, and 1 cm had been regarded as 1,000 units. The largest measurement recorded had been about 830 units, and any measurement not repeating within ± 4 had been viewed with suspicion, with the exception that occasionally, in the very largest measurements, a variation of ± 10 had been accepted, but then only when it was consistent with the remainder, which had all been within ± 4 . In the smaller measurements, allowable differences were reduced to 1 or 2; hand and eye could get no nearer than that.

In answer to Dr Zienkiewicz, gravity stresses had been calculated on free slices and not by "tumbler" methods, since arch action, owing to the method of construction of the dam, could not affect the results. The stresses induced by temperature changes, estimated on the basis of encastré arch theory, were given in Paper No. 6155.*

Professor Pippard endorsed what Professor Allen had said about the Authors' gratitude to the consulting engineers and their clients for the opportunity to carry out the investigation; it had been well worth while from many different points of view.

Dr Jaeger had suggested that for many years experts in dam design had considered that a relaxation investigation should be attempted. That was news to Professor Pippard who, when he had originally mentioned the possibility to Mr Binnie, had rather felt he

* See p. 259.

was suggesting an operation of a new type and without any assurance that it could be carried through successfully.

Perhaps what Dr Jaeger really meant was that a three-dimensional solution of the elastic equation by any possible means was very desirable, but until now it had been, Professor Pippard thought, unobtainable except possibly for certain very artificial and idealized shapes.

Professor Pippard had the impression that Mr Morgan felt that because the assumptions which it was necessary to make might not in reality be completely satisfied, since concrete was not uniform from point to point, it did not matter if doubtful methods of analysis were used to determine the stresses. Professor Pippard appreciated Mr Morgan's point and agreed that the answer, when it had been obtained, should not be taken for more than its worth, but the addition of an extremely doubtful method of computation to doubtful assumptions would confuse the issue. Certain idealized assumptions had had to be made for the "mathematician's dam" which it was known were not strictly complied with in practice. One of those assumptions had been that the structure behaved perfectly elastically, which it did not. Calculations on such assumptions, however, had provided a basis of comparison which was very valuable. To what extent such results represented the real stresses would depend on the conditions in the actual dam and that emphasized the need, not only in the dam problem but in many other problems, for correlation between theory and full-scale-test results.

In the case of the Dokan dam a good deal of exploratory work had been done to determine whether or not it would be possible to carry out full-scale tests during construction. Instruments could certainly be installed to measure something which had the dimensions of a strain, but how much of that strain would be due to the stress in the concrete, how much to creep, how much to temperature, how much to water movements, and how much to other factors, there had seemed to be no way of assessing. It appeared to Professor Pippard that many of the results deduced from full-scale strain measurements were of even more doubtful validity than most of the methods proposed for calculating the stresses.

Several speakers had referred to approximate methods of calculation. Professor Pippard did not want to discuss that at the moment, because the Authors were studying such methods closely and had, for example, found that they were able to deduce the main stresses on the centre section of the dam directly from the model measurements*: they had obtained results which compared reasonably with the relaxation calculations. Professor Allen's three-dimensional stress analysis was valuable in providing a yard-stick for comparison and made it possible to determine the degree of accuracy of a particular approximate method. That would probably prove to be one of the most valuable results of the research.

Dr Zienkiewicz had raised the question of measuring strains directly in the model: that had not been done for the simple reason that it was not easy to design a satisfactory extensometer. When the work had started they had thought that they would be able to plaster the model with resistance strain-gauges, but on soft rubber, which had a Young's modulus of 100 to 120 lb/sq. in., the resistance gauge had been much too stiff.

Taking that into consideration, the Authors had decided to rely on displacements plotted over the whole surface of the dam, and fortunately the resultant curves had been remarkably smooth and had given the information needed.

References had been made in the discussion to the effects of Poisson's ratio, and the Authors had been asked whether or not those were small. It was necessary to define what effects were meant: when a piece of indiarubber had been placed in a triaxial testing machine and hydrostatic pressure applied, no observable change in dimensions took place; i.e., rubber had proved incompressible and Poisson's ratio was therefore $\frac{1}{2}$. Another material having the same value for E might show a very big strain under the same stress. His view was that although the effect on strains might be quite serious when the value

* See Paper No. 6155, p. 259.

of Poisson's ratio was altered, the stresses were probably not seriously modified. The matter, however, required much more research and a series of mathematical and experimental investigations were being made at Imperial College to throw further light on that issue.

Professor Pippard felt that the technique of frozen-stress photoelasticity had not reached the stage where it would be likely to produce convincing results in such a complex problem as that of a large arch dam, as suggested by Mr Christodoulides.

REFERENCES

11. Carlo Semenza, "The Most Recent Dams by the 'Società Adriatica di Eletticità (S.A.D.E.)' in the Eastern Alps." *Proc. Instn Civ. Engrs*, Part I, vol. 1, p. 508 (Sept. 1952).
12. L. W. Atcherley and Karl Pearson, "On Some Disregarded Points in the Stability of Masonry Dams." University of London, *Drapers' Company Research Memoirs*, Technical Series II, 1904. (Abstract in *Min. Proc. Instn Civ. Engrs*, vol. 162, p. 456.)
13. J. S. Wilson and W. Gore, "Stresses in Dams: an Experimental Investigation by means of India-Rubber Models." *Min. Proc. Instn Civ. Engrs*, vol. 172 (1907-08), p. 107.
14. A. J. S. Pippard, "The Functions of Engineering Research in the University" (Unwin Memorial Lecture, 1949). *J. Instn Civ. Engrs*, vol. 33, p. 265 (Feb. 1950).
15. L. F. Richardson, *Phil. Trans. Roy. Soc., London*, vol. 210 (1909).
16. L. Fox and R. V. Southwell. *Phil. Trans. Roy. Soc., C*, pp. 1151-56 (1941); *A*, p. 239, pp. 419-60 (1945).
17. O. C. Zienkiewicz, "The Stress-Distribution in Gravity Dams." *J. Instn Civ. Engrs*, vol. 27, p. 244 (Jan. 1947).
18. A. J. S. Pippard and Letitia Chitty, "Some Problems presented by Cable Bracing." *Structural Paper No. 10. Instn Civ. Engrs*, 1945.
19. A. Stucky, "Etude sur les barrages arqués" ("Study of arch dams"). *Bull. Tech. Suisse Romande*, vol. 48 (1922), p. 1 (7 Jan.); p. 25 (4 Feb.); p. 49 (4 Mar.); p. 85 (15 Apr.); p. 97 (29 Apr.).
20. William Cain, "The Circular Arch under Normal Loads." *Proc. A. S. Civ. Engrs*, vol. 47 (1921), p. 285.
21. A. W. Hendry, "Photoelastic Experiments on the Stress-Distribution in a Diamond Head Buttress Dam." *Proc. Instn Civ. Engrs*, Part I, vol. 3, p. 370.
22. *Loc. cit.*, *Correspondence*: vol. 4, p. 101 (Jan. 1955).
23. H. Spooner and L. D. McConnell, "An Ethoxylene resin for Photoelastic Work." *Brit. Appl. Phys*, vol. 4, p. 181 (June 1953).
24. M. M. Frocht and R. Quernsey, "A Special Investigation to develop a General Method for Three-dimensional Photoelastic Stress Analysis." *Nat. Adv. Ctee Aero.*, Technical Note 2822, Washington, Dec. 1950.
25. S. P. Christodoulides, "A Photoelastic Investigation into End Anchorages of Prestressed Concrete Beams." (Ph.D. Thesis submitted to the University of London, 1955.)
26. M. Clutterbuck, "A Photoelastic Investigation of the Effect of Poisson's ratio on some Two- and Three-dimensional Stress Distributions." (M.Sc. Thesis submitted to the University of London, 1955.)
27. R. D. Hoyle, "Temperature Stresses in Irregular Solids." *Nature*, vol. 157 (1951), p. 30.

CORRESPONDENCE on the foregoing Paper is closed and no contribution, other than those already received at the Institution, will now be accepted.—SEC.

Paper No. 6155

THE DETERMINATION OF THE STRESSES IN AN ARCH DAM FROM A RUBBER MODEL

by

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and

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(Ordered by the Council to be published with written discussion)

SYNOPSIS

On pp. 198 *et seq* the Authors, with Professor Allen and Dr Severn, described an analysis of the Dokan dam by a combination of mathematical and experimental methods. This treatment, different from any hitherto attempted, provides a solution of the elastic equations to any required degree of accuracy, but is lengthy and requires a skilled team.

Reference was there made to the possibility of calculating approximate values of the most important stresses, and certain promising methods have since been further developed. The present Paper describes some of these and deals particularly with one for deducing the main values of the vertical and hoop stresses due to water pressure from measurements of radial and tangential displacements on a rubber model of the dam. Such a direct estimate is possible, despite the difference in values of Poisson's ratio for concrete and rubber, by correcting the measured displacements on the basis outlined in the first Paper. The results now obtained are sufficiently near to those found by relaxation analysis to provide an adequate check on design; they give considerable promise of providing reasonably quick estimates of maximum stresses in future designs of arch dams. The vertical stresses are calculated by fitting a formula to the corrected radial displacements measured on the central vertical section of the model, and from it deducing the bending moments at any horizontal section. The hoop stresses are found simply by a formula derived from the elastic equations for the central section.

Certain other lines of approach are also described and approximations to gravity and temperature stresses are given.

INTRODUCTION

IN a recent Paper¹ the Authors and others described an investigation of the stresses in an arch dam under construction at Dokan.

The investigation was based on relaxation analysis supplemented by experiments on rubber models, and is perhaps the most complete solution of the full elastic equations for a structure of this type yet undertaken. Reference was made to attempts in progress to estimate the main stresses by approximate methods and now that a mathematical solution is available, it is clearly important to use it as a standard of comparison, to evolve simpler methods for use in design of future dams.

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¹ The references are given on p. 275.

Not only is the complete relaxation calculation very long but it requires skilled technicians and it would generally be impracticable except as a final check on the design of a large and important structure.

The present Paper is concerned mainly with results achieved so far in the calculation of stresses by a direct use of the experimental results from rubber models. Other lines of approach are of course possible, but the field of research is very large; although both experimental and analytical investigations are being actively continued, it must be some time before it is possible to say definitely in which direction the most promising solution will be found. In the meantime the Paper records results sufficiently hopeful to provide at least a possible interim method for preliminary design.

To avoid reproduction of the elaborate curves of model results and the diagrams of stress-distribution, etc., determined by the complete relaxation analysis, reference should be made to the earlier Paper (see p. 198).

A developed section along the water-face of the Dokan dam is shown in *Fig. 2*, p. 199, indicating the grid reference system used in the earlier work, whilst the profile of maximum depth is shown in *Fig. 5*, p. 206. The thickness of the profile in metres at any depth z metres below the crest is:

$$d = 6.2 + 0.13z - 0.0003314z^2 + 0.000018045z^3 \quad . \quad . \quad (1)$$

From this expression the figures given in *Fig. 5*, p. 206 were calculated.

It will be observed from *Fig. 2*, p. 199 that throughout the lower portion of the Dokan valley symmetry was good, but at the upper levels there were appreciable departures. Despite this, however, displacement curves for the model indicated that there was, near the middle of the portion of the dam of greatest height, a plane about which the elastic behaviour of the structure was reasonably symmetrical.

The importance of symmetry has been correctly pointed out by many designers, and without further investigation it would be unwise to assume that all the methods described in this Paper, which have proved reasonably satisfactory when applied to the Dokan dam, can be used safely for structures in valleys which are markedly unsymmetrical.

GRAVITY STRESSES

An arch dam is generally built of a number of separate monoliths, the vertical joints between which are finally grouted to form the arch structure. The gravity stresses, arising from the dead-weight, are thus unaffected by the arch action, and can be calculated with the same degree of accuracy as for a gravity dam, by the engineer's usual method, assuming linear stress distribution across horizontal sections. The great accuracy of this method has been shown by Zienkiewicz,² who solved the elastic equations for a gravity dam by relaxation methods, and his conclusions were confirmed during the study of the Dokan dam. Table 1 compares the results obtained by relaxation analysis of the Dokan profile and by the engineer's usual method; the agreement leaves no doubt as to the reliability of the latter treatment.

STRESSES DUE TO WATER PRESSURE

Since gravity stresses can be so accurately calculated by well-known methods, the main problem of the arch dam is the determination of stresses arising from water pressure on the back; in this Paper attention is directed largely to finding the values

TABLE 1.—VERTICAL STRESSES DUE TO GRAVITY
(Measured in lb/sq. in.; tension positive)

Grid	Engineer's theory		Relaxation	
	Water face	Air face	Water face	Air face
0	0	0	0	0
2	-53	-22	-45	-30
4	-110	-30	-100	-40
6	-163	-30	-155	-45
8	-215	-20	-205	-40
10	-265	-3	-260	-10
12	-303	13	-300	30
14	-338	30	-340	50

of these stresses at the centre of the dam which, from the full analysis, is known to be the section governing design.

As described previously,¹ the first attempt made by relaxation methods was a simplified plane-stress solution, which became known for convenience as the "tumbler," since it was based on the assumption that the dam was part of a complete cylindrical tube having a wall of the same profile as the maximum section of the dam, firmly held at the base and subjected to the full head of water. Every vertical section of the tumbler was similar and similarly loaded, and it was hoped that the great simplification of the equations thus obtained would result in a practical solution of sufficient accuracy. Unfortunately, the time required to carry out the relaxation analysis even on these simplified assumptions was considerable. The results, shown in Table 2, do not compare favourably in an important respect with those from the

TABLE 2.—STRESSES FROM TUMBLER AND COMPLETE RELAXATION
ANALYSES

Combined water and gravity loading (measured in lb/sq. in.;
tension positive)

Face	Grid	Vertical stress		Hoop stress	
		Tumbler	Relaxation	Tumbler	Relaxation
Water	0	0	0	-200	-290
	2	-110	-110	-295	-380
	4	-260	-240	-370	-460
	6	-355	-335	-400	-480
	8	-370	-340	-365	-450
	10	-315	-280	-285	-380
	12	-215	-160	-190	-280
	14	-45	-60	-85	-170
Air	0	0	0	-215	-190
	2	30	20	-295	-200
	4	130	100	-330	-200
	6	155	120	-360	-180
	8	140	100	-340	-140
	10	20	30	-310	-90
	12	-130	-140	-240	-50
	14	-250	-300	-180	-20

complete solution: the tumbler gave a good picture of the vertical but not the hoop stresses.

Other methods were therefore tried, based for the most part on the results of experiments on a rubber model.

Before describing these, reference must be made to a discussion¹ on the effects upon the behaviour of the structure arising from a variation in the value of Poisson's ratio. It was indicated there that the radial displacements of the air face of the dam might be expected to be almost independent of Poisson's ratio, and that the change in thickness of a concrete dam at any section would be much less, proportionately, than in a rubber model. Indeed, for practical purposes the radial displacements of the water and air faces on any horizontal radial line in the real dam could be assumed to be equal; this deduction was confirmed by the complete relaxation calculations. Furthermore, it was found that if an equivalent value of Young's modulus equal to $7/6$ times that derived from a simple bending test was used, the full-scale values of radial displacements deduced from measurements on the rubber model were very close indeed to those determined by relaxation calculations. When this modified value of E was used, the full-scale tangential displacements deduced from the model also agreed well with those calculated, but vertical displacements so deduced were considerably bigger. These results were entirely consistent with the effects expected from the difference in Poisson's ratio.

In deducing stresses directly from the rubber model, therefore, by methods now to be described, it was decided to use a value of E modified as stated above, i.e., 140 lb/sq. in. instead of the estimated value of 120 lb/sq. in., and furthermore to avoid, if possible, any use of the measured vertical displacements. The latter restriction means that different methods have been used for the calculation of the vertical (zz) and the hoop ($\theta\theta$) stresses.

The correction applied to E is not altogether arbitrary, but can be justified to some extent qualitatively because the strains measured in any direction in a body under tri-axial stress are related to the stresses causing them by a function involving both Young's modulus and Poisson's ratio. When, as usually occurs in the dam, all three stresses are compressive, the strains in any direction are less than those which would occur from a uni-axial stress in that direction alone; in other words, the equivalent E is larger.

Polynomial solution for vertical (zz) stresses

In this method the largest set of radial air-face displacements u_z measured at a number of different depths z on the section of the model nearest to that about which it exhibited elastic symmetry, were fitted to a polynomial expression of the form:

$$u_z' = A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \dots \quad (2)$$

as many terms being taken as found necessary.

Appropriate numerical values of the constants A, B, C , etc., had to be determined. Some of these were found from the physical conditions of the problem and the remainder by applying a least-square method to discrepancies between measured displacements and the corresponding values found from equation (2). Suppose, for example, that u_n' is the particular value of u_z' from equation (2) at a depth $z = n$, and u_n is the experimental value at the same point. Then if $e_n = u_n' - u_n$, the condition of least squares requires that the constants shall have such values as will make $\sum e_n^2$ a minimum, i.e.:

$$0 = \frac{\partial \Sigma e_n^2}{\partial A} = \frac{\partial \Sigma e_n^2}{\partial B} = \frac{\partial \Sigma e_n^2}{\partial C} = \dots = \frac{\partial \Sigma e_n^2}{\partial F}. \quad (3)$$

Now, $e_n^2 = (A + Bn + Cn^2 + \dots)^2 - 2u_n(A + Bn + Cn^2 + \dots) + u_n^2$, and the relations (3) become

$$\left. \begin{aligned} 0 &= \Sigma A + B \Sigma n + C \Sigma n^2 + D \Sigma n^3 + E \Sigma n^4 + F \Sigma n^5 - \Sigma u_n, \\ 0 &= A \Sigma n + B \Sigma n^2 + C \Sigma n^3 + D \Sigma n^4 + E \Sigma n^5 + F \Sigma n^6 + \Sigma u_n n, \\ 0 &= A \Sigma n^2 + B \Sigma n^3 + C \Sigma n^4 + D \Sigma n^5 + E \Sigma n^6 + F \Sigma n^7 - \Sigma u_n n^2, \\ 0 &= A \Sigma n^3 + B \Sigma n^4 + C \Sigma n^5 + D \Sigma n^6 + E \Sigma n^7 + F \Sigma n^8 - \Sigma u_n n^3, \\ 0 &= A \Sigma n^4 + B \Sigma n^5 + C \Sigma n^6 + D \Sigma n^7 + E \Sigma n^8 + F \Sigma n^9 - \Sigma u_n n^4, \\ 0 &= A \Sigma n^5 + B \Sigma n^6 + C \Sigma n^7 + D \Sigma n^8 + E \Sigma n^9 + F \Sigma n^{10} - \Sigma u_n n^5 \end{aligned} \right\} \quad (4)$$

This method has the great advantage that the number of simultaneous equations to be solved depends only on the number of constants to be evaluated, i.e., on the number of terms in the polynomial, and not on the number of experimental points included; when the constants are substituted in (2) there results an equation for the displacements of the air face of the model dam at the centre section. From the general argument outlined earlier, these displacements depend very little upon the value of Poisson's ratio for the material of the model and it may be argued therefore that if this ratio for rubber had been 0.15, i.e., the same as for concrete, instead of its true value of 0.50, the equation would have been little altered. Furthermore, although in the model there were large differences between the radial displacements at the air and water faces of the dam, the same argument, confirmed by relaxation analysis, indicates that if Poisson's ratio had in fact been 0.15 these differences would have been reduced to negligible amounts. The equation may therefore, without serious error, be assumed to apply both to the air face and to the water face of the model, on the assumption that Poisson's ratio is the same as for concrete. The only corrections then necessary to deduce values for the actual dam are those for Young's modulus and for scale. Both these are straightforward, and the multiplying factor to obtain true dam displacements from the equation is $n^2 E_m / E_c$. The value of E_m was taken as 140 lb/sq. in. for reasons already discussed, E_c was 2,000,000 lb/sq. in., as assumed in the relaxation analysis, and the scale of the model, $1/n$, was $1/252$. The conversion factor is then 4.45.

TABLE 3.—COMPARISON OF MEASURED DISPLACEMENTS u WITH VALUES OF u' DERIVED FROM POLYNOMIAL

Grid	u measured	u' from polynomial: cm $\times 10^{-3}$		
		Last term z^4	Last term z^5	Last term z^6
0	-795	-795	-795	-795
2	-824	-824.2	-829.4	-828.2
4	-831	-824.0	-829.0	-828.7
6	-782	-776.9	-777.1	-778.0
8	-668	-678.1	-673.6	-674.2
10	-533	-535.9	-531.1	-530.5
12	-372	-371.3	-372.1	-371.4
14	-225	-218.2	-224.9	-226.1
16	-120	-123.3	-120.1	-120.2

On the assumption that deflexions are due entirely to bending, and that stresses are distributed linearly across any section, the bending moment M_z and stress f_z at the two faces at a depth z can then be calculated from:

$$M_z = E_c I_z \frac{d^2 u'}{dz^2} \quad \text{and} \quad f_z = \mp \frac{1}{2} E_c d_z \frac{d^2 u'}{dz^2}, \quad . \quad . \quad . \quad (5)$$

where d_z denotes the width of the dam. The upper sign relates to the water face, the lower to the air face.

In fact some of the displacement results from shear, but investigation showed that corrections on stress from this were so small as to fall within the assigned limits of error; they were therefore neglected.

The shearing force S_z and the intensity of loading p_z causing bending on the vertical strip at the centre of the dam may also be estimated respectively from:

$$S_z = \frac{d}{dz} \left(E_c I_z \frac{d^2 u'}{dz^2} \right) \quad \text{and} \quad p_z = \frac{d^2}{dz^2} \left(E_c I_z \frac{d^2 u'}{dz^2} \right) \quad . \quad . \quad . \quad (6)$$

The unit for z was taken as two grids, which was the spacing in the model experiments. The origin for the polynomial was first taken in the base at grid 22, the assumption

being that at that level $u' = \frac{du'}{dz} = 0$. This implied that both the constants A and B were zero and the first two of equations (4) were not used. These assumptions do not, however, necessarily lead to zero value of bending moment at the crest, although in fact it chanced that when terms up to Fz^5 were included, the bending moment there was very small.

In later calculations, the origin was taken at the crest; it was assumed that the value of u_0 was correct, and that the bending moment at $z = 0$ was zero. The first and third of equations (4) were not then needed, since A was experimentally known and C was necessarily zero. The assumption that u_0 was correct is justifiable, since this particular value was checked many times during the experiments. Had it, however, been felt undesirable to assume this, the first of equations (4) could have been included to determine the constant A .

Table 3 shows the experimental values of u as measured on the model, together with corresponding values of u' obtained by the method described when terms in equation (2) so far as z^4 , z^5 , and z^6 , respectively, were included. It is clear from these results that nothing was gained by including terms beyond z^5 ; this form was therefore adopted.

By using the conversion factor of 4.45 already mentioned, the polynomial gave values applicable to the full-scale dam. The full-scale stresses were then determined from equation (5). The results are shown in Table 4 compared with those by relaxation, when gravity stresses are included.

Although the divergence is considerable at certain sections on the water face the maximum compressive stress occurs at level 8, whether calculated directly from the model results or by relaxation, the values being respectively -376 lb/sq. in. and -340 lb/sq. in. On the air face the maximum tension is at level 6, that deduced from the model being 146 lb/sq. in. against 120 lb/sq. in. by relaxation.

When the analysis was carried further, by the use of equation (6), it was found that at a level approximately 5.6 grids down from the crest, the loading on the vertical strip changed sign. Above this level the loading was opposite in direction to the water load, rising to a maximum of 3.8 lb/sq. in. at the crest, where there was also a shear of 1,740 lb. Too much reliance should not be placed on the exact values

TABLE 4.—VERTICAL STRESSES FROM POLYNOMIAL AND RELAXATION
Combined water and gravity loading (measured in lb/sq. in.;
tension positive)

Grid	Polynomial values		Relaxation values	
	Water face	Air face	Water face	Air face
0	0	0	0	0
2	-135	60	-110	20
4	-255	115	-240	100
6	-339	146	-335	120
8	-376	141	-340	100
10	-350	82	-280	30
12	-258	-59	-160	-140
14	-8	-300	-60	-300

quoted, since they involve the second, third, and fourth differential of the original polynomial; they do serve, however, to indicate that if the dam is regarded as a set of interacting arches and cantilevers, at all events on the central section, the arches above grid 5.6 carry more than the water load, whilst below that level they carry progressively less. The top arch, which may be regarded as that portion of the structure 5 m above water level, in addition to the loading of 3.8 lb/sq. in., also carries the shear; these together are roughly equivalent to a total loading of 12.6 lb/sq. in.

It should be noted that neither in this, nor in the next section, is any assumption made about the distribution of loading round the arches, but the above results are quoted since they are consistent with results given later when assumptions on load distribution on elementary arches are made.

Direct deduction of hoop ($\theta\theta$) stresses from model

The stress-strain relations at any point in the dam are:

$$\left. \begin{aligned} \theta\theta - \sigma(\widehat{rr} + \widehat{zz}) &= \frac{E}{r} \left(u + \frac{\partial w}{\partial \theta} \right) \\ \widehat{zz} - \sigma(\widehat{rr} + \theta\theta) &= E \frac{\partial v}{\partial z} \\ \widehat{rr} - \sigma(\theta\theta + \widehat{zz}) &= E \frac{\partial u}{\partial r} \end{aligned} \right\} \dots \dots \dots (7)$$

The second of (7) contains a derivative of v , the vertical displacement which, as already shown, is particularly affected by a variation of Poisson's ratio and is therefore to be avoided when deducing full-scale stresses. The first and third of these equations may be combined to obtain:

$$(\theta\theta - \widehat{rr})(1 + \sigma) = E \left\{ \frac{1}{r} \left(u + \frac{\partial w}{\partial \theta} \right) - \frac{\partial u}{\partial r} \right\} \dots \dots \dots (8)$$

On the water face of the dam $\widehat{rr} = -wh$, the water pressure. On the curved air face $\widehat{rr}$ is not strictly zero, but it is a sufficiently good approximation to take it as such. Since, as already explained, the water and air face values of u can be taken as the

same when Poisson's ratio is 0.15, it may be assumed that $\frac{\partial u}{\partial r} = 0$. At the centre section the values of $\frac{\partial w}{\partial \theta}$ are well defined, both on the water and the air face, as can be seen from the curves¹ of w . Using the experimental air face values of u and δ the appropriate measured $\frac{\partial w}{\partial \theta}$ together with the scale factor 4.45, the stresses were found from equation (8) with $E = 2,000,000$ lb/sq. in. and $\sigma = 0.15$. They are given in Table 5 compared with the values from the complete relaxation analysis.

TABLE 5.—HOOP STRESSES BY DIRECT DEDUCTION FROM MODEL, FROM RELAXATION, AND FROM RING THEORY
(Measured in lb/sq. in.; tension positive)

Grid	From model		Relaxation		Ring theory
	Water face	Air face	Water face	Air face	
0	-290	-150	-290	-190	0
2	-370	-190	-380	-200	-162
4	-430	-220	-460	-200	-357
6	-450	-220	-480	-180	-478
8	-420	-180	-450	-140	-542
10	-370	-130	-380	-90	-562
12	-290	-60	-280	-50	-552
14	-230	-20	-170	-20	-524

The correlation between these two sets of results is very good. All stresses are compressive and the maximum values both on the water and air faces occur at the same level by either method. The greatest hoop stress deduced from the model is 450 lb/sq. in. compared with 480 lb/sq. in. by relaxation analysis. This difference is insignificant.

Equation (8) can be used to obtain $\widehat{\theta\theta}$ at any point on the dam, but at the abutments the experimental values of $\frac{\partial w}{\partial \theta}$ are not so well defined since, when the present experiments were made, it was not foreseen that these particular values might be required. In further studies now in progress on idealized valleys, more measurements close to the abutments will be taken, to determine whether accurate values of $\frac{\partial w}{\partial \theta}$ are practicable.

When the data at present available are used for the determination of the hoop stresses at the abutments, they are found to be rather lower than the relaxation values.

Determination of hoop stresses by ring theory

The values for the hoop stresses in Dokan based on the simple formula $\widehat{\theta\theta} = -whR/d$, where wh denotes the water pressure, R the radius to the water face, and d the width of the section, have been included in the last column of Table 5. This formula assumes that the dam is formed by a pile of circular rings placed one on top of another, but in no way interacting. Each ring is free to contract under the

water pressure acting at that level. It is interesting to note the maximum figure of -562 lb/sq. in. for Dokan, since dams have in the past been designed on this theory. In particular, a series of arch dams built in Australia³ at the beginning of this century were all designed on this basis.

Determination of hoop stresses from an analysis of arch displacements

When making a preliminary estimate of the hoop stresses arising from water load in an arch dam, it is usual to imagine the structure to be split into a series of horizontal arches, each acting separately and carrying uniform intensity of loading. In an attempt to determine how far this assumption was applicable to Dokan, an analysis was made of the deflexions recorded at various horizontal levels round the model.

It was observed that the measured radial and tangential displacements were reasonably symmetrical about a radial axis approximately $1\frac{1}{2}^\circ$ from the experimental station 0, towards the left bank of the valley, i.e., 48° from the left bank and 67° from the right bank. This axis was therefore chosen for the zero grid in the relaxation calculations.

Taking this axis as the downstream direction, the air-face radial displacements (for reasons already discussed) and the mean of the water and air face tangential displacements were combined into upstream and cross-stream components. When plotted, these revealed certain rigid-body movements, almost entirely downstream, together with a very small spread in the upper arches. Fig. 1 illustrates the analytical procedure in relation to the arch at grid level 2, where the broken line shows the reflexion of the displacements in the left half-arch to compare directly with those in the right half.

Each half-arch was examined separately, those towards the left bank as though they were halves of symmetrical arches of included angle $2 \times 48^\circ$, those towards the right bank as though they were halves of symmetrical arches of included angle $2 \times 67^\circ$.

Having deducted the rigid-body movements and allowed for the spread, the modified components were recombined into modified radial displacements, also shown in Fig. 1, and tangential displacements.

Suppose now that a circular arch rib carries a uniform radial load of intensity p , as in Fig. 2. Any elastic rotations of the abutments A and B which reduce the encastré moments there will be clockwise at A and anti-clockwise at B. Provided that the width of the arch is less than a certain value which depends on the radius and the included angle of the arch, it can be shown that such relief at the abutments will result in a rather smaller radial deflexion at the centre. Since the curve of radial displacements for the encastré arch has not only a zero value at each abutment but also a zero slope it will be appreciated that the deflexion curves for encastré and relieved arches must cross somewhere in each half span. By considering the energy absorbed in bending and compression (but neglecting that due to shear), it can be shown that the point at which they cross is independent of the degree of relief, the point being common to all such deflexion curves. If the arch is thicker than the limiting value, the encastré curve lies everywhere inside that for the relieved arch. This property was used in an alternative analysis of the upper arches of Dokan, which satisfy the criterion down to grid 8.

General formulae for radial and tangential displacements of arches carrying uniform radial loading, and having specified end conditions, were developed, in terms of pR_0^2/Ed , where p denotes the radial loading, R_0 the mean radius of the arch, d the width, and E Young's modulus. Thus the appropriate curves for radial

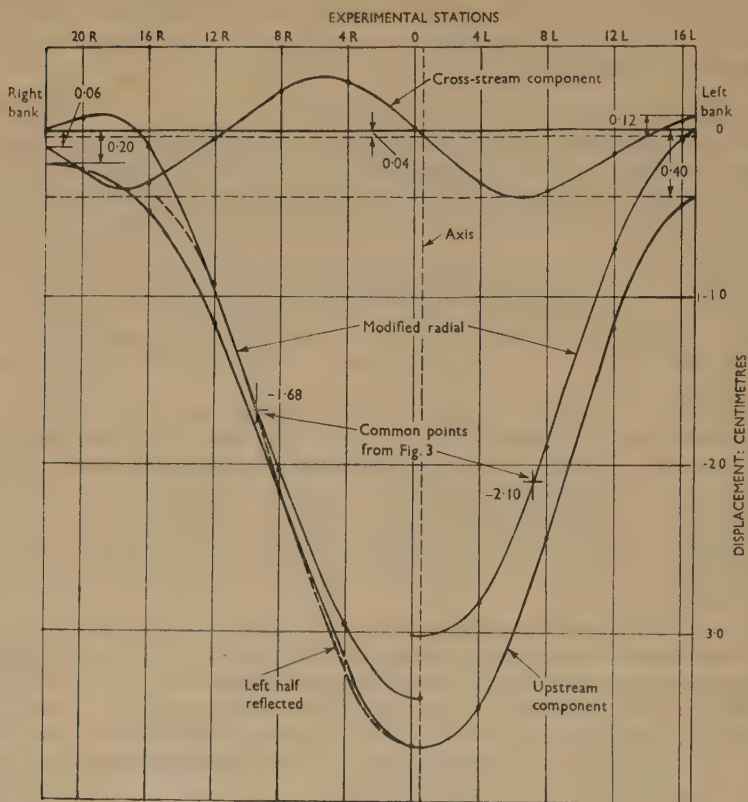


FIG. 1.—ANALYSIS OF DISPLACEMENTS AT GRID LEVEL 2

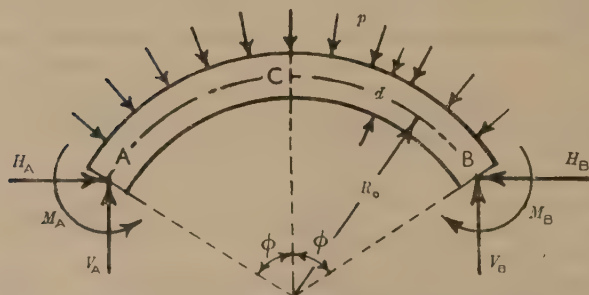


FIG. 2.—ARCH UNDER UNIFORM RADIAL PRESSURE

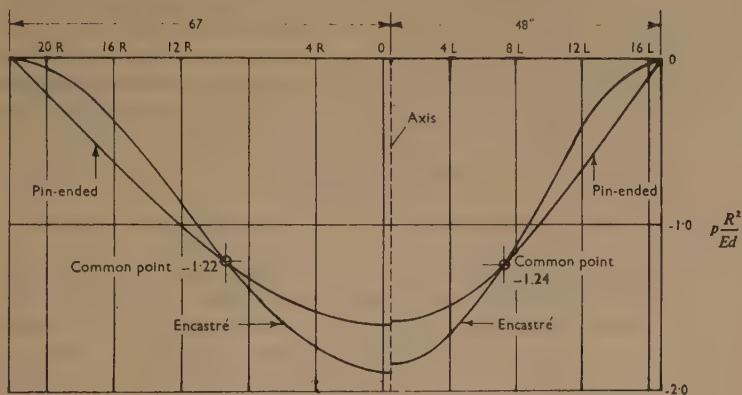


FIG. 3.—RADIAL DISPLACEMENT CURVES SHOWING COMMON POINTS

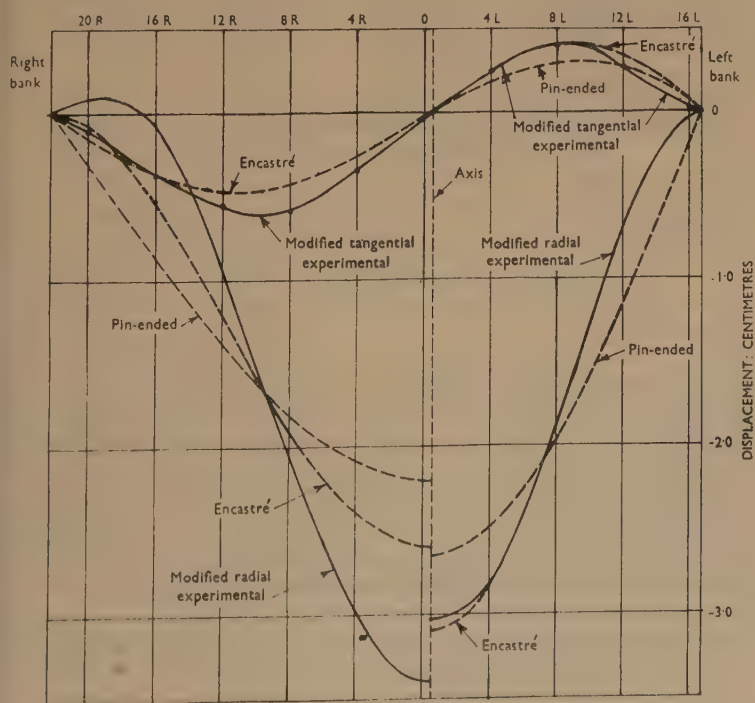


FIG. 4.—COMPARISON OF EXPERIMENTAL WITH THEORETICAL CURVES OF DISPLACEMENT

displacements corresponding to encastré and pin-end conditions for the particular arch considered could be drawn, as shown in Fig. 3, where a common point is indicated at $-1.24 pR_0^2/Ed$.

The modified radial displacement curve already described was then plotted to such a scale as to make it pass through this common point, as in Fig. 4, in order to compare its shape with that for uniform radial loading. The scale required determines the intensity of radial loading. Thus, referring to Figs 1 and 3,

$$-2.10 = -1.24 pR_0^2/Ed$$

whence

$$p = 19.6 \text{ lb/sq. in.}$$

The modified tangential curve was then drawn to the same scale, in order to compare with the corresponding curves for encastré and pin-ended arches, also shown in Fig. 4.

In the analysis of the half-arches towards the left bank, with the exception of that above top water level, each modified radial curve lay everywhere between the pin-end and the encastré curve but nearer to the latter (in the arch illustrated they are practically coincident), thus providing some justification for the usual assumption that they are encastré and carry uniform loading. The curves of tangential displacements provided further evidence of agreement, except near the abutment, where the experimental values are smaller. In the top arch, the curve of radial displacements lay beyond the encastré curve, indicating a positive rotation at the abutment. This rotation, and also the small spread, have been allowed for in the estimate for the stresses given in Table 6, but the allowable reduction in stress from the relieving rotation in arches 4 and 6 has been neglected.

TABLE 6.—ANALYSIS OF EXPERIMENTAL DISPLACEMENTS AND DEDUCED CENTRAL HOOP STRESSES COMPARED WITH RELAXATION, FOR LEFT HALF OF DAM
(Tension positive)

Grid	Body displacement: cm		Spread: cm	Rotation: min.	Loading: lb/sq.in.	Hoop stresses: lb/sq. in.			
	Down	Across				Deduced		Relaxation	
						Water face	Air face	Water face	Air face
0	0.49	0.06	0.28	0.45	11.3	—296	—121	—290	—190
2	0.40	0.04	0.12		19.6	—388	—176	—380	—200
4	0.30				28.1	—461	—189	—460	—200
6	0.20				35.2	—493	—157	—480	—180

In the analysis of the half-arches towards the right bank, each modified curve of radial displacement lay beyond the corresponding encastré curve, but both the loadings and the stresses were lower than those deduced for the left half.

This analysis also gives excellent agreement with the relaxation results, the maximum values on the water face at level 6 being -493 and -480 lb/sq. in. respectively, and on the air face -189 and -200 lb/sq. in.

Analysis based on central portion of the model

A research student⁴ concentrated attention upon the central portion of the dam of approximately constant, maximum height, between grid numbers 9R and 9L.

The measured displacements of points on both the surfaces, rotations of the centre, and of the vertical boundary sections of this portion. The movements at these boundaries were naturally much larger than those at the abutments and their values could therefore be obtained with correspondingly greater accuracy. By the time these measurements were made the rubber had stiffened considerably, the shape of the displacement curves indicating that the effect was in the nature of a progressive increase in the effective value of E_m . The procedure adopted was therefore to fit a polynomial curve to the central radial displacements in the manner already described, then to modify this by an approximate small allowance for the displacements due to shear (see p. 264). The loading curve for the central vertical strip, deduced

TABLE 7.—SOLUTION BASED ON CENTRAL, PARALLEL PORTION OF THE MODEL
COMPARED WITH RELAXATION
(Tension positive)

Face	Grid	Vertical stress: lb/sq. in.		Hoop stress: lb/sq. in.	
		Model	Relaxation	Model	Relaxation
Water	0	0	0	-282	-290
	2	-122	-110	-300	-380
	4	-260	-240	-462	-460
	6	-340	-335	-505	-480
	8	-336	-340	-535	-450
	10	-270	-280	-519	-380
	12	-157	-160	-263	-280
	14	-30	-60		-170
Air	0	0	0	-116	-190
	2	47	20	-179	-200
	4	124	100	-230	-200
	6	152	120	-236	-180
	8	105	100	-190	-140
	10	3	30	-29	-90
	12	-144	-140		-50
	14	-300	-300		-20

from this modified polynomial, was then plotted to determine where the loading was zero, i.e., at what level the whole of the water load was carried by arch action. On the assumption that an elementary arch rib at this level carried a uniform radial load, and was subject to the measured end conditions, the central deflexion was evaluated in terms of p/E_m . Since p was, for this particular level, the full water load, equation of the measured and the theoretical deflexions enabled the effective value of E_m to be determined. The full-scale zz stresses were then deduced from the modified polynomial, and the $\theta\theta$ or hoop stresses on the assumption that all the arches were uniformly loaded. Table 7 gives the stresses thus calculated.

The vertical stresses deduced by this method show excellent agreement with the relaxation values. The hoop stresses are not quite so good but the maximum value of -535 lb/sq. in. from the model is less than 12% higher than the maximum relaxation value of -480 lb/sq. in.

It is interesting to note that if the method just described for determining the effective Young's modulus for the model from the position of zero loading on the

central strip, is applied to the set of experimental results used earlier to illustrate the polynomial analysis of displacements, the value deduced is approximately 145 lb/sq. in.; this is in close agreement with the value 140 lb/sq. in. adopted as a result of quite different arguments.

STRESSES DUE TO TEMPERATURE CHANGES

Model experiments of the kind described are of no use to estimate the effects of a rise or fall in temperature and direct calculation of a conventional type is necessary. The dam was considered to consist of a number of independent horizontal encastré arches and the temperature was assumed to have reached a steady state and to vary linearly through the thickness, at any level, from the value at the air face to that at the water face. On this understanding, suppose that a rise of t_i° at the air face accompanies a rise of t_e° at the water face. For the purpose of calculation this can be replaced by two superposed systems, one consisting of a uniform rise $t = \frac{1}{2}(t_i + t_e)$ throughout the thickness, the other of a linear variation from a rise $t' = \frac{1}{2}(t_i - t_e)$ at the air face to a drop t' (or rise $-t'$), at the water face.

Under a uniform rise t , if the arch was free to expand, it would spread by an amount $\alpha t R_0 \sin \phi$ at each abutment, α being the coefficient of thermal expansion for the constructional material. This expansion is, however, prevented and compressive and bending hoop stresses are consequently induced which can be calculated from the following formulae. At the centre:

$$\left. \begin{aligned} f_c &= -\frac{E\alpha t}{3} \cdot \frac{d^2}{R_0^2} \cdot \frac{\phi \sin \phi}{\phi F}, \\ f_b &= \pm 2E\alpha t \frac{d}{R_0} \cdot \frac{\sin \phi (\phi - \sin \phi)}{\phi F}, \end{aligned} \right\} \quad \dots (9)$$

where

$$\phi F = \left(1 + \frac{d^2}{12R_0^2}\right) \phi (2\phi + \sin 2\phi) - 4 \sin^2 \phi$$

and the upper sign relates to the water face, and the lower to the air face. At the abutments

$$\left. \begin{aligned} f'_c &= f_c \cos \phi, \\ f'_b &= -f_b \frac{(\sin \phi - \phi \cos \phi)}{\phi - \sin \phi} \end{aligned} \right\} \quad \dots (10)$$

The second system, i.e., the steady gradient from t' at the air face to $-t'$ at the water face induces a pure couple, from which $f_b = \pm E\alpha t'$ at every section of the arch.

Variations of air temperature at Dokan are very similar to those at Boulder Dam, and *Fig. 29a*, p. 222, shows the expected mean temperatures whilst *Fig. 29b*, p. 222, gives the estimated maximum departures from these means. It was assumed that grouting of the joints will take place at 5°F below the mean temperatures, and on this understanding Table 8 shows the maximum extra stresses likely to be induced by a rise or fall in temperature.

These values for the stresses are admittedly overestimates since they take no account of the restraint from cantilever action or other alleviating factors but they give a picture of the general effect, and show that the combined effects of a constant rise and an equal rise and fall on the air and water faces respectively are cumulative at the centre of the dam, but tend to annul each other at the abutments. The

TABLE 8.—TEMPERATURE STRESSES IN ENCASTRÉ ARCHES GROUTED AT 5°F
BELOW MEAN TEMPERATURES
(Tension positive)

Face	Grid	Temperature rise: °F	Hoop stress: lb/sq. in.		Temperature fall: °F	Hoop stress: lb/sq. in.	
			Centre	Abutment		Centre	Abutment
Water	0	27.5 - 2.5	63	-71	-17.5 + 2.5	-55	36
	2	23.2 - 6.8	117	-35	-13.2 + 6.8	-77	-9
	4	21.9 - 8.1	137	-37	-11.9 + 8.1	-111	-16
	6	20.6 - 9.4	161	-55	-10.6 + 9.4	-128	-17
	8	19.3 - 10.7	194	-100	-9.3 + 10.7	-149	-3
	10	17.9 - 12.1	221	-150	-7.9 + 12.1	-166	20
	12	17.5 - 12.5	186	-270	-7.5 + 12.5	-151	50
	14	17.5 - 12.5	128	-281	-7.5 + 12.5	-126	50
Air	0	27.5 + 2.5	-77	69	-17.5 - 2.5	57	-34
	2	23.2 + 6.8	-125	31	-13.2 - 6.8	101	12
	4	21.9 + 8.1	-147	31	-11.9 - 8.1	117	20
	6	20.6 + 9.4	-179	45	-10.6 - 9.4	138	23
	8	19.3 + 10.7	-230	75	-9.3 - 10.7	167	15
	10	17.9 + 12.1	-315	90	-7.9 - 12.1	208	10
	12	17.5 + 12.5	-398	100	-7.5 - 12.5	243	25
	14	17.5 + 12.5	-410	35	-7.5 - 12.5	248	56

TABLE 9.—HOOP STRESSES AT CENTRAL SECTION AT MAXIMUM AND MINIMUM
TEMPERATURES

Combined temperature and water loading
(Tension positive)

Face	Grid	Maximum temperature		Minimum temperature	
		Approx. lb/sq. in.	Relaxation lb/sq. in.	Approx. lb/sq. in.	Relaxation lb/sq. in.
Water	0	-230	-290	-340	-290
	2	-250	-375	-440	-380
	4	-290	-440	-540	-470
	6	-290	-430	-580	-515
	8	-220	-370	-570	-505
	10	-150	-270	-530	-450
	12	-100	-175	-440	-350
	14	-100	-80	-350	-230
Air	0	-230	-290	-100	-130
	2	-310	-330	-90	-110
	4	-370	-380	-100	-80
	6	-400	-385	-90	-45
	8	-410	-360	-20	0
	10	-450	-320	70	60
	12	-460	-290	180	110
	14	-420	-285	230	155
	sill		-275		185

results for combined water load and temperature at the centre section are shown in Table 9 where they are compared with the relaxation results.

CONCLUSIONS

Quite apart from the quality of the material used in construction, the maximum calculated stresses allowable in a structure must depend on the methods used to predict them. Thus, if the method is known to give greater stresses than will actually be developed, higher limits will be tolerated. Bourgin,⁵ in a discussion on methods of design, comment as follows on early dams designed by the simple ring, or cylinder, theory: "Experience has shown that these dams have stood safely and that the cylinder theory gives the value of the average factor of safety of the structure." He then points out that, theoretically, the stresses in an encastré arch are about twice those in the corresponding ring similarly loaded, and that Italian engineers, who design on a basis of independent horizontal encastré arches, are therefore prepared to accept half the factor of safety obtained by cylinder theory.

There is a wide gap between the very elementary ring theory and the elaborate trial-load method developed by the U.S.A. Bureau of Reclamation, or the relaxation process used for Dokan. These last two methods seek to give a picture of the stresses actually set up in an ideal elastic structure under the given conditions. The relaxation process probably furnishes the most nearly exact solution yet produced for the mathematicians' dam, since it solves, to any required degree of accuracy, the fundamental elastic equations which govern the stress distribution both in the dam and in the surrounding rock.

This being so, it is interesting to note that, judged by the standard of relaxation results, the ring theory does, in fact, give a good estimate of the *factor of safety in compression* for Dokan, since the maximum stress developed according to ring theory is -562 lb/sq. in. against -515 lb/sq. in. according to relaxation (the latter stress being developed only under extreme conditions of low temperature). But the ring theory gives no indication even of the existence of tension in the structure.

The results of the work described in this Paper may be summarized as follows:—

Table 1 indicates that the ordinary Engineer's theory gives an excellent picture of both the compressive and tensile stresses developed under gravity alone.

Table 2 indicates that a two-dimensional solution by relaxation, whilst giving a fair picture of the vertical stresses, gives a distorted estimate of the hoop stresses.

If stresses are to be deduced from a rubber model, certain simple adjustments must be made to allow not only for scale and difference in Young's modulus, but also to allow for difference in Poisson's ratio. When these adjustments are made, the displacements measured on the model can be interpreted as full-scale displacements on the actual dam. Table 4 then indicates that the polynomial solution for vertical stresses, based on radial displacements measured on the central section of the rubber model, provides a fair picture of the maximum vertical stresses due to water load. In essence, the analytical process consists in setting up four simultaneous equations, and solving them. Table 5 indicates, further, that a fair picture of the hoop stresses at the central section, due to water load, can be obtained from the model by direct deduction, a straightforward and easily applied method. These are the two methods which so far seem most hopeful for use in design. Unfortunately, when the measurements were made on the model, insufficient evidence was collected in the neighbourhood of the abutments to enable an accurate prediction of the hoop stresses developed there to be made in the same way.

Two other methods for analysing the model displacements are discussed, one based on the properties of arches, the other on the behaviour of an isolated portion of the dam, in this instance, the portion of the structure of practically constant maximum height. Each contains devices which may prove useful, but both are open to criticism in that assumptions of a doubtful nature have to be made, at some stage, as to the distribution of loading on elementary horizontal arches. Both these methods give fair estimates for the stresses developed in the upper part of the Dokan dam, as will be seen from Tables 6 and 7.

All these results go to show that a moulded-rubber model can be used successfully to predict the full-scale stresses, quite independently of relaxation analysis. The expense of such a model is small in relation to the importance of the objective.

The rubber model can give no help about stresses induced by changes in temperature, but a simple analytical method, based on the behaviour of independent encastré arches, is suggested for predicting the extra hoop stresses developed. As might be expected, this method overestimates the effects of temperature changes, but it shows the general trend, and affords a reasonably quick method for investigating where the critical hoop stresses, both compressive and tensile, are likely to occur. When combined with the stresses deduced from the model, given in Table 5, the greatest compressive and tensile stresses are found to occur under conditions of lowest temperatures, and in the same positions as predicted by relaxation (see Table 9).

ACKNOWLEDGEMENTS

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REFERENCES

1. D. N. de G. Allen, Letitia Chitty, A. J. S. Pippard, and R. T. Severn, "Experimental and Mathematical Analysis of Arch Dams, with special reference to Dokan." *Proc. Instn Civ. Engrs*, Pt I, vol. 5, p. 198 (May 1956).
2. O. C. Zienkiewicz, "The Stress-Distribution in Gravity Dams." *J. Instn Civ. Engrs*, vol. 27, p. 224 (Jan. 1947).
3. L. A. B. Wade, "Concrete and Masonry Dam-Construction in New South Wales." *Min. Proc. J. Instn Civ. Engrs*, vol. 178 (1908-09), p. 1.
4. P. G. Valvis, Unpublished thesis for the degree of M.Sc. University of London, 1954.
5. A. Bourgin, "The Design of Dams." Pitman, London, 1953. See p. 163.

The Paper, which was received on 19 January, 1956, is accompanied by four sheets of drawings from which the Figures in the text have been prepared.

CORRESPONDENCE on this Paper should be forwarded to reach the Institution before 15 September, 1956. Contributions should not exceed 1,200 words.—SEC.

Paper No. 6107

SHEET-PILE WALLS AT FAILURE

by

*** Peter Walter Rowe, Ph.D., A.M.I.C.E.***(Ordered by the Council to be published with written discussion)*

SYNOPSIS

Experiments are described with model walls in loose sand where the walls failed in bending or penetration. The results are compared with Brinch Hansen's limit design method and the Author's flexibility method, and indicate that either method may be used as an initial basis for limit design.

Passive pressure occurs above the tie level for naturally yielding tie-rods and driven-and-dredged walls, maintaining the tie load at or slightly above the free-earth-support value, depending on tie yield, tie depth, and wall yield. The "limit" at which the wall is considered to have failed is the first yield point, and the problem is strictly one of deformation. Thereafter failure is associated with backfill subsidence, causing triangular active pressure distribution at ultimate collapse for all initial conditions. Contrary to Brinch Hansen, yield is accompanied by increasing moments.

Modified soil constants may be necessary for use in Brinch Hansen's method with compressible subsoils. The principles of the method are compared with the flexibility method, and the latter is shown to be of general application to subsoils of all compressibilities. Vertical thrust influences the stability of pinned steel piles in bearing and encasté piles at yield.

INTRODUCTION

SHEET-PILE walls may be designed to a certain factor of safety on either the first yield point or the ultimate strength of the anchorage, wall section, and earth resistance beneath the dredge level. In the first case an elastic type of working stress design has been proposed^{1, 2} based on a knowledge of the relative flexibility of the pile and the subsoil. In the second case a plastic-type limit design has been proposed by Brinch Hansen³ based on the assumption that the wall at failure develops yield hinges and that the soil is in a state of failure. Pile flexibility and soil compressibility do not enter directly into the calculations.

Both methods of design are of value to a proper understanding of the factors governing the stability of this type of structure. No experimental data have been available, however, with which to compare the actual and theoretical failure states and for this reason experiments have been made on walls designed to fail in bending or penetration. In all cases the anchorage was designed not to fail.

BRINCH HANSEN'S LIMIT DESIGN METHOD

Brinch Hansen's method has been fully described³ and applied to particular wall designs. However, it is necessary to obtain a general solution for walls at any scale

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¹ The references are given on p. 314.

in order to study the validity of the principles of the method by comparison with the model behaviour. For this reason the equations are derived for a uniform dry sand on each side of the piling. The equations are of academic interest only and therefore do not appear in Brinch Hansen's work, to which the reader should refer for practical design purposes. A brief outline of the basis of the calculations is given below.

The possible states of failure, assuming a stable anchorage, are stated to depend on penetration depth as shown in Figs 1a to e. The wall (Fig. 1a) may be designed just not to yield under the bending moment acting at passive failure of the subsoil. With slightly increased penetration, passive failure and wall failure may be arranged simultaneously (Fig. 1b). With increasing penetration the wall beneath the dredge

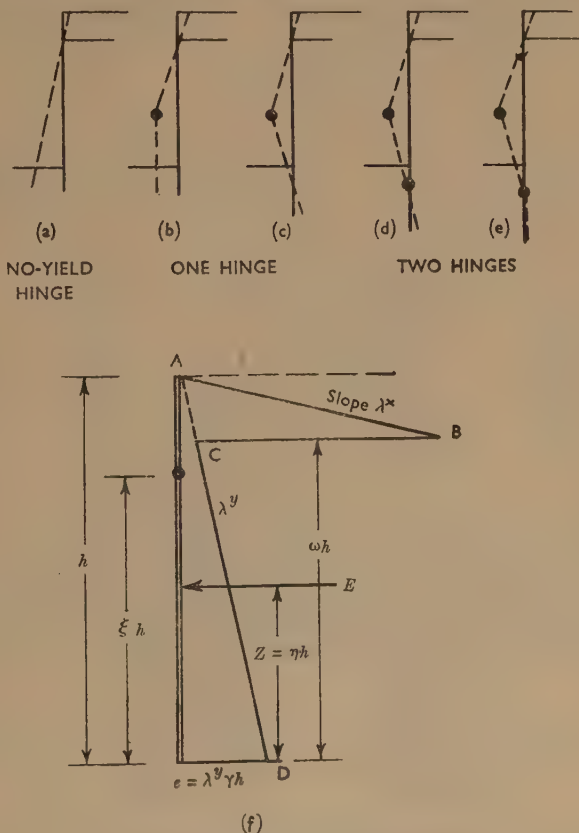


FIG. 1.—THEORETICAL FAILURE STATES

level is in a state of rotation when yield occurs (Fig. 1c) and a second yield hinge finally develops beneath the dredge level (Figs 1d and e).

The general solution for the total earth force and its point of application on a stiff wall rotating about any non-yielding pin (Fig. 1f) has been obtained and provides the basis for the analysis of the equilibrium of the composite rotating walls shown in

Figs 1a to e. However, in order to determine the maximum bending moment on the wall the distribution of earth pressure is required. Since the soil adjacent to the wall is not necessarily in a state of failure the pressure distribution cannot be calculated by failure theory. Therefore, the distribution is assumed to be given by two triangular distributions, AB and CD joined by a pressure jump CB. The slope of the line AB is taken as the maximum passive pressure coefficient for a wall rotating into the soil about the pin and the coefficients governing the position of the pressure jump and the slope of the line CD are calculated such that the distribution gives the same force E and point of application Z on the wall. The coefficients λ , η , λ^x , λ^y and ω give all the values required for a given rotation point ξ and these have been calculated for $\phi = \pm 30^\circ$ and plotted in graphs for positive and negative rotation. Extracts from these graphs for use in the following calculations are given in Table 1.

TABLE 1

ξ	λ		η	λ^x		λ^y		ω
	Positive rotation	Negative rotation		Positive rotation	Negative rotation	Positive rotation	Negative rotation	
5.5	0.266	5.16	0.3333	5.66	—	—	—	1.0
1.0	—	5.0	0.310	—	—	—	—	—
0.9	0.284	—	0.455	—	—	0.225	—	0.898
0.8	0.293	—	0.478	—	—	0.221	—	0.889
0.7	0.302	—	0.500	—	—	0.215	—	0.878
0.64	0.316	—	0.524	—	—	0.211	—	0.864
0.60	0.328	—	0.538	—	—	0.213	—	0.856
0.57	0.340	—	0.545	—	—	0.215	—	0.850
0	0.350	—	0.553	—	—	0.218	—	0.842
0	5.66	0.266	0.333	5.66	0.266	1.70	4.40	0

The calculations are made for types (a), (b), and (e) states of failure, as shown in Figs 1 with a uniform soil of density γ . The notation is that used by Brinch Hansen with the addition of the coefficients α and β (Fig. 4b). The final results are given in terms of the Author's previous notation (Fig. 4b).

The following group of equations apply in part to all the calculations:

$$\left. \begin{aligned} E_1 &= \frac{1}{2}\gamma\lambda_1 h_1^2 & Z_1 &= \eta_1 h_1 \\ E_2 &= \frac{1}{2}\gamma\lambda_2 h_2^2 & Z_2 &= \eta_2 h_2 \\ E_3 &= \frac{1}{2}\gamma\lambda_3 h_3^2 & Z_3 &= \eta_3 h_3 \\ E_4 &= \frac{1}{2}h_4(e_3 + e_4) & E_4 Z_4 &= \frac{1}{8}\gamma h_4^2(2e_3 + e_4) \\ e_3 &= \gamma h_3 \lambda_3^y & e_4 &= \gamma h_1 \lambda_4 \end{aligned} \right\} \quad . \quad . \quad (1)$$

PINNED WALLS

Design with no-yield hinge and passive failure

In this design (Fig. 2a): equating forces:

$$A = E_1 - E_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Taking moments about the toe:

$$Aq - E_1 Z_1 + E_3 Z_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

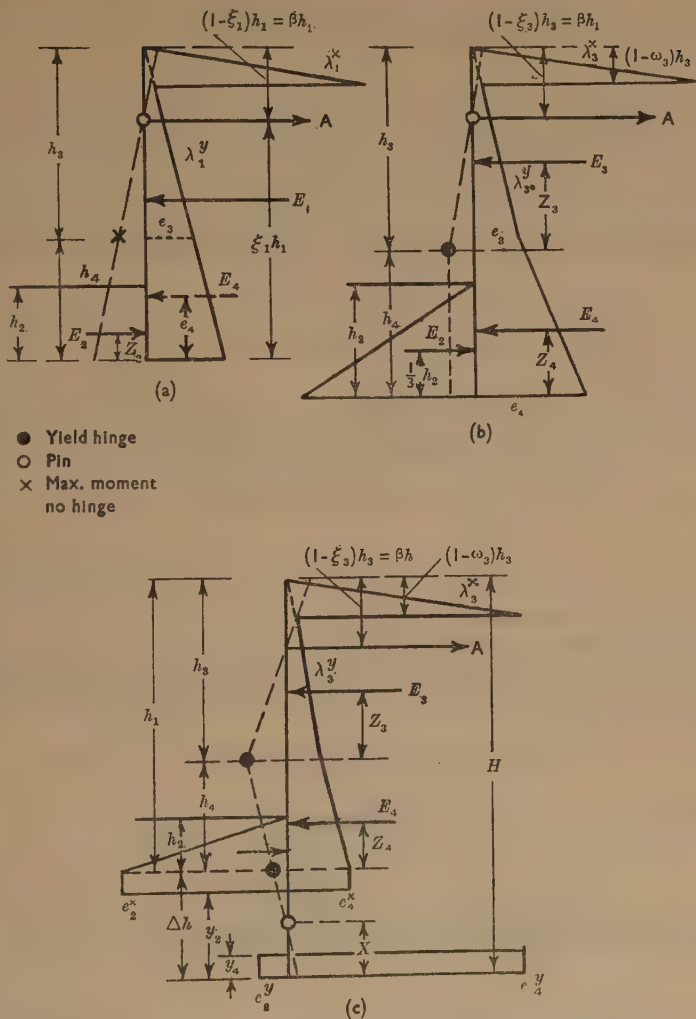


FIG. 2.—PINNED WALLS AT FAILURE

Substituting for A , E_1 , E_2 , Z_1 , Z_2 from equations (1) and (2) in equation (3) and writing $q = h_1(1 - \beta)$ the following equation may be solved for $\left(\frac{h_2}{h_1}\right) \equiv (1 - \alpha)$:

$$(1 - \beta) \left[\lambda - \lambda_2 \left(\frac{h_2}{h_1} \right)^2 \right] - \lambda_1 \eta_1 + \lambda_2 \eta_2 \left(\frac{h_2}{h_1} \right)^3 = 0 \quad (4)$$

Choosing a series of values of β , 0 to 0.3, the corresponding values of $\xi_1 = (1 - \beta)$ are used to select the correct values of the coefficients λ and η from Table 1.

The maximum bending moment M_2 occurs at depth h_3 where the shear force is zero. Therefore, equating the forces acting below h_3 :

$$E_2 = E_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The active pressure coefficients λ_3^y and λ_4^y in equation (1) are both equal to λ_1^y in this case. Substituting equations (1) in equation (5):

$$\frac{h_3}{h_1} = \sqrt{\left(1 - \frac{\lambda_2}{\lambda_1^y} \left(\frac{h_2}{h_1}\right)^2\right)} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Taking moments about the point of maximum moment for pressures below the point:

$$M_2 = E_4 Z_4 - E_2 Z_2$$

whence

$$\frac{M}{\gamma H^3} \equiv \frac{M_2}{\gamma h_1^3} = \frac{\lambda_1^y}{6} \left(1 - \left(\frac{h_3}{h_1}\right)^2\right) \left(2\left(\frac{h_3}{h_1}\right) + 1\right) - \frac{\lambda_2}{2} \eta_2 \left(\frac{h_2}{h_1}\right)^3 \quad . \quad . \quad (7)$$

Substituting equations (1) in equation (2):

$$\frac{T}{\gamma H^2} \equiv \frac{A}{\gamma h_1^2} = \frac{1}{2} \left[\lambda_1 - \lambda_2 \left(\frac{h_3}{h_1}\right)^2 \right] \quad . \quad . \quad . \quad . \quad (8)$$

The values of α , $\frac{M}{\gamma H^3}$, and $\frac{T}{\gamma H^2}$, all in dimensionless units, are plotted in Figs 4a, 5a, and 5b.

Design with one yield hinge and passive failure

For this design (Fig. 2b) a yield hinge is chosen at depth h_3 . The pressure diagram above this is that for the pinned wall Fig. 1f. The length h_4 is to translate into the soil and the pressure e_4 is taken as that at the base of a wall height h_1 . Thus $\xi_2 = \xi_4 = \infty$.

Equating forces above and below the hinge respectively:

$$A = E_3; \quad E_2 = E_4 \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Substituting values of E_2 and E_4 from equation (1) in equation (9):

$$h_2 = \sqrt{\frac{(1 - h_3/h_1)(\lambda_3^y h^3/h_1 + \lambda_4)}{\lambda_2}} \quad . \quad . \quad . \quad . \quad (10)$$

Taking moments about the hinge, above and below the hinge:

$$M_2 = E_3(h_3 - Z_3 - (h_1 - q)) \quad . \quad . \quad . \quad . \quad (11)$$

$$M_2 = E_4 Z_4 - E_2 Z_2 \quad . \quad . \quad . \quad . \quad (12)$$

Equating equations (11) and (12), substituting for all values from equation (1) and h_2 equation (10), writing $\frac{h_1 - q}{h_3} = (1 - \xi_3)$, and rearranging:

$$3\lambda_3 \left(\frac{h_3}{h_1}\right)^3 (\xi_3 - \eta_3) + \frac{1}{\lambda_2^{\frac{1}{2}}} \left[\left(1 - \frac{h_3}{h_1}\right) \left(\frac{h_3}{h_1} \lambda_3^y + \lambda_4\right) \right]^{3/2} - \left(1 - \left(\frac{h_3}{h_1}\right)^2\right) \left[2\lambda_3^y \left(\frac{h_3}{h_1}\right) + \lambda_4 \right] = 0 \quad . \quad . \quad . \quad (13)$$

Substituting values from equation (1) in equations (9) and (11):

$$\frac{T}{\gamma H^2} = \frac{A}{\gamma h_1^2} = \frac{\lambda_3}{2} \left(\frac{h_3}{h_1}\right)^2 \quad . \quad . \quad . \quad . \quad (14)$$

$$\frac{M}{\gamma H^3} = \frac{M_2}{\gamma h_1^3} = \frac{\lambda_3}{2} \left(\frac{h_3}{h_1} \right)^3 (\xi_3 - \eta_3) \quad . \quad . \quad . \quad (15)$$

and

$$\alpha = \left(1 - \frac{h_2}{h_1} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The solution of equations (13), (10), (14), and (15) for a *particular* tie level must proceed by trial and error, since the value of h_3 and hence ξ_3 is unknown at first for tie levels below the top. However, the general equations are solved directly by choosing a series of ξ_3 values which together with the values in Table 1 give all the solutions. The corresponding tie level coefficients β are then found from the relation:

$$\beta = (1 - \xi_3) \left(\frac{h_3}{h_1} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Design with two yield hinges

With this design (Fig. 2c) the penetration is determined to be consistent with a second yield hinge at depth h_2 below the dredge level, where the bending moment M_3 will be equal in value and opposite in sign to M_2 . The equilibrium above and below the lower hinge is considered separately.

Above the lower hinge

Equations (9), (10), and (11) apply. Taking moments about the upper hinge for forces between the hinges:

$$E_4 Z_4 - E_2 Z_2 = M_2 - M_3 = 2M_2 \quad . \quad . \quad . \quad (18)$$

Substituting M_2 equation (11) and values from equation (1) into equation (18) and rearranging:

$$6\lambda_3 \left(\frac{h_3}{h_1} \right)^3 (\xi_3 - \eta_3) + \frac{1}{\lambda_2^{1/2}} \left[\left(1 - \frac{h_3}{h_1} \right) \left(\frac{h_3}{h_1} \lambda_3 \gamma + \lambda_4 \right) \right]^{3/2} - \left(1 - \frac{h_3}{h_1} \right)^2 \left[2 \left(\frac{h_3}{h_1} \lambda_3 \gamma + \lambda_4 \right) \right] = 0 \quad . \quad . \quad . \quad (19)$$

The values of $\frac{A}{\gamma h_1^2}$ and $\frac{M_2}{\gamma h_1^3}$ are given by equations (14) and (15).

The wall heights h_2 , h_4 , rotate about the base, so that $\xi_2 = \xi_4 = 0$. A series of solutions for equations (19), (10), (14), and (15) are obtained taking an arbitrary series of ξ_3 values.

Below the lower hinge

The additional penetration depth Δh is calculated using an approximate pressure diagram (Fig. 2c). The length Δh is treated as if perfectly stiff, rotating about a point distance X from the toe.

The pressures used are given by the following identities:

$$\left. \begin{aligned} e_4^x &= \gamma h_1 \lambda_1^x; & e_2^x &= \gamma h_2 \lambda_2^x \\ e_4^y &= \gamma h_1 \gamma_1^y; & e_2^y &= \gamma h_2 \lambda_2^y \end{aligned} \right\} \quad . \quad . \quad . \quad (20)$$

The exact value of ξ for wall height Δh is unknown at first, and the calculation proceeds taking coefficients for the case of $\xi = 0$. From Table 1:

$$\left. \begin{aligned} \Delta e_1^x &= e_2^x - e_4^x = (5.66h_2 - 0.266)\gamma \\ \Delta e^y &= e_4^y - e_2^y = (4.40 - 1.70h_2)\gamma \end{aligned} \right\} \quad . \quad . \quad . \quad (21)$$

Equating forces:

$$y_4 \Delta e^y = (\Delta h - y_2) \Delta e^x \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Taking moments about the hinge:

$$-M_3 = y_4 \Delta e^y \left(\Delta h - \frac{y_4}{2} \right) - \Delta e^x \frac{(\Delta h - y_2)^2}{2} \quad . \quad . \quad . \quad (23)$$

For small rotations about the point distance X from the toe the following approximate ratios are given:

$$\frac{y_4}{X} = 0.5 \quad \frac{y_2}{X} = 1.8 \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Substituting for Δe^x , Δe^y equations (21), and for y_4 and y_2 equations (24), into equation (22):

$$\frac{\Delta h}{X} = \frac{9.34h_2 + 1.72}{5.66h_2 - 0.266} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Substituting values from equations (21), (24), and (25) into equation (23) and writing $M_2 = -M_3$, with rearrangement:

$$\frac{\left(\frac{h_1}{\Delta h} \right)^2 = \frac{(4.4 - 1.7h_2)}{4 \left(\frac{M_2}{\gamma h_1^3} \right)} \cdot \frac{(16.7h_2 + 1.38)(5.66h_2 - 0.266)}{(9.34h_2 + 1.72)^2} \quad . \quad . \quad (26)$$

whence

$$\left. \begin{aligned} H &= h_1 + \Delta h \\ \alpha &= \frac{1 - h_2/h_1}{1 + \Delta h/h_1} \\ \frac{T}{\gamma H^2} &= \frac{1}{(1 + \Delta h/h_1)^2} \times \frac{A}{\gamma h_1^2} \\ \text{and} \quad \frac{M}{\gamma H^3} &= \frac{1}{\left(1 + \left(\frac{\Delta h}{h_1} \right) \right)^3} \times \frac{M}{\gamma h_1^3} \end{aligned} \right\} \quad . \quad . \quad . \quad (27)$$

The solutions of equations (19), (10), (14), (15), and (26) are sufficient to solve equations (27).

Negative moments at the tie level

In all the above calculations it is assumed that no yield hinge occurs at the tie level. If such a hinge was to develop, no further increase in passive resistance above the tie would occur. Therefore the calculations are valid only up to that tie level at which the negative moment at the tie M_t equals the maximum positive moment, where uniform pile sections are used.

Taking moments about the tie for pressures above the tie, the following equations are obtained.

Case (1). Tie level above the active pressure jump

$$\frac{M_t}{\gamma H^3} = 0.943\beta^3 \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Case (2). Tie level below the active pressure jump

(a) No yield hinge

$$\frac{M_t}{\gamma H^3} = (\lambda_1^x - \lambda_1^y) \left(\frac{1 - \omega}{2} \right)^2 \left(\beta - \frac{2}{3} (1 - \omega) \right) + \frac{\lambda_1^y}{6} \beta^3 \quad . \quad . \quad (29)$$

(b) *One or two yield hinges*

$$\frac{M_t}{\gamma H^3} = (\lambda_3 x - \lambda_3 y) \left(\frac{1 - \omega}{2} \right)^2 \left(\frac{h_3}{H} \right)^2 \left(\beta - \frac{2}{3}(1 - \omega) \frac{h_3}{H} \right) + \frac{\lambda_3 y}{6} \cdot \beta^3 \quad (30)$$

The graphs $\frac{M_t}{\gamma H^3} / \beta$ are plotted in Fig. 5b.

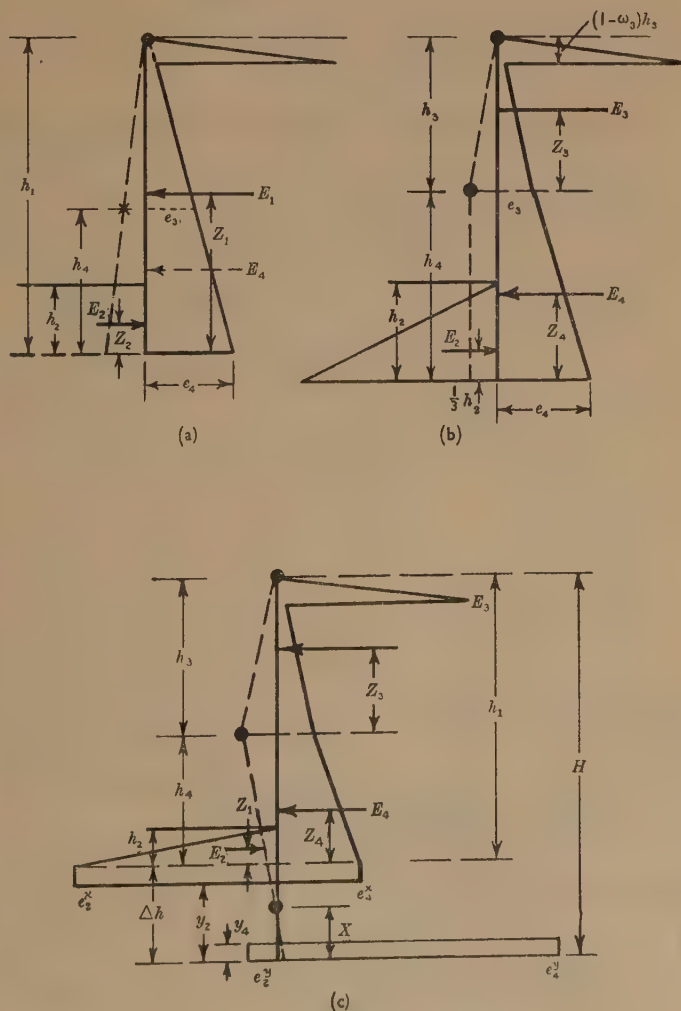


FIG. 3.—ENCASTRÉ WALLS AT FAILURE

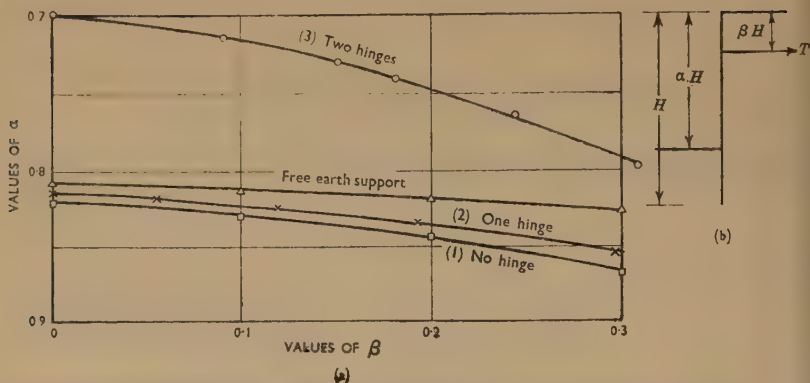


FIG. 4.—THEORETICAL PENETRATION COEFFICIENTS FOR LOOSE SAND. $\phi = 30^\circ$

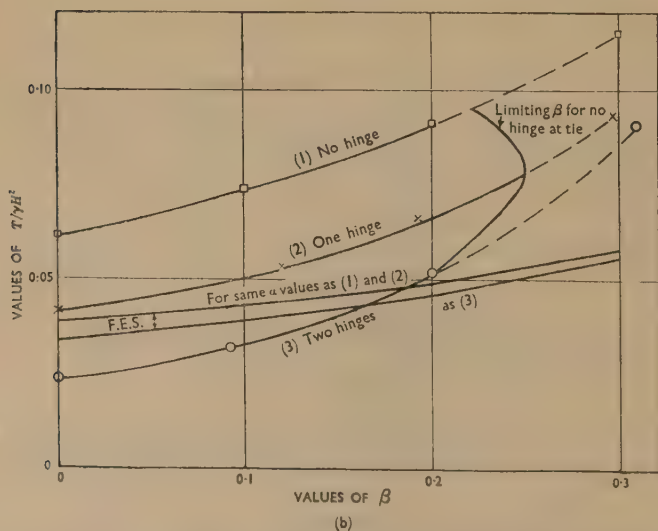
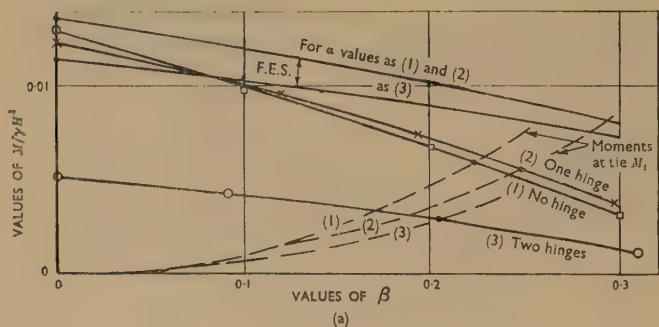


FIG. 5.—BENDING-MOMENT AND TIE-LOAD LIMIT VALUES FOR $\phi = 30^\circ$

WALLS ENCASTRÉ AT THE ANCHORAGE

For the model tests, the retained sand was always placed level with the anchorage.

One yield hinge and passive failure

The economical design for this case (Fig. 3a), is the one where the maximum positive moment M_2 is just equal to the maximum negative moment M_1 . By adjusting the factors of safety so that the hinge at the top forms first, the calculation proceeds with the whole wall assumed to be infinitely stiff and rotating about the anchorage.

Therefore

$$\xi_1 = 1.0; \quad \xi_2 = \frac{h_1}{h_2}$$

Inserting

$$\lambda_3 v = \lambda_4 = \lambda_1 v \text{ in equation (10):}$$

$$h_2 = \sqrt{\frac{\lambda_1 v}{\lambda_2} \left(1 - \left(\frac{h_3}{h_1} \right)^2 \right)} \quad . \quad . \quad . \quad (31)$$

Equating moments about the anchorage and the point at depth h_3 :

$$-M_1 = M_2$$

Therefore

$$h_1(E_1 - E_2) - E_1 Z_1 + E_2 Z_2 = E_4 Z_4 - E_2 Z_2 \quad . \quad . \quad . \quad (32)$$

Substituting values from equations (1) and (31) in equation (32):

$$\frac{\lambda_1}{\lambda_1 v} (1 - \eta_1) + \left(1 - \left(\frac{h_3}{h_1} \right)^2 \right) \left[\frac{2}{3} \sqrt{\frac{\lambda_1 v}{\lambda_2} \left\{ 1 - \left(\frac{h_3}{h_1} \right)^2 \right\}} - 1 \right] - \frac{1}{3} \left(1 - \left(\frac{h_3}{h_1} \right)^2 \right) \left(2 \left(\frac{h_3}{h_1} \right) + 1 \right) = 0 \quad . \quad . \quad . \quad (33)$$

Using equations (32) and (2):

$$\left. \begin{aligned} \frac{M}{\gamma H^3} &= \frac{M_2}{\gamma h_1^3} = \frac{1}{6} \left[\lambda_1 v \left(1 - \left(\frac{h_3}{h_1} \right)^2 \right) \left(2 \frac{h_3}{h_1} + 1 \right) - \lambda_2 \left(\frac{h_2}{h_1} \right)^3 \right] \\ \text{and} \quad \frac{T}{\gamma H^2} &= \frac{A}{\gamma h_1^2} = \frac{1}{2} \left[\lambda_1 - \lambda_2 \left(\frac{h_2}{h_1} \right)^2 \right] \end{aligned} \right\} \quad . \quad . \quad (34)$$

Equations (33), (31), and (34) are solved for the one case only taking:

$$\xi_2 \approx 5.5, \text{ whence } \alpha = 0.83; \quad \frac{T}{\gamma H^2} = 0.07; \quad \frac{M}{\gamma H^3} = 0.0094.$$

Two yield hinges and passive failure

In this case (Fig. 3b) a second hinge is assumed to form at the point of maximum positive moment so that the wall height h_3 rotates about the top and $\xi_3 = 1.0$. The lower length translates into the soil and $\xi_2 = \xi_4 = \infty$. Taking moments about the anchorage for height h_3 :

$$M_2 - M_1 = E_3(h_3 - Z_3) \quad . \quad . \quad . \quad (35)$$

and

$$M_2 = -M_1 \quad . \quad . \quad . \quad (36)$$

The remaining basic equations are given by equations (9) and (12).

Substituting for M_1 equation (36) and M_2 equation (12) together with values from equation (1) and h_2 from equation (10) in equation (35) gives:

$$\frac{3}{2} \lambda_2 \left(\frac{h_3}{h_1} \right)^3 (1 - \eta_3) + \frac{1}{\lambda_2^{1/2}} \left[\left(1 - \left(\frac{h_3}{h_1} \right) \left(\lambda_3 v \frac{h_3}{h_1} + \lambda_4 \right) \right)^{3/2} - \left(1 - \left(\frac{h_3}{h_1} \right)^2 \right) \times \left(2 \lambda_3 v \frac{h_3}{h_1} + \lambda_4 \right) \right] = 0 \quad . \quad . \quad (37)$$

These equations give:

$$\alpha = 0.826; \quad \frac{T}{\gamma H^2} = 0.0535; \quad \frac{M}{\gamma H^3} = 0.00895.$$

Three yield hinges

The calculation for this (Fig. 3c) is similar to that for two hinges on a pinned wall where $\xi_3 = 1$; $\xi_2 = \xi_4 = 0$.

Equating forces above the bottom hinge gives equation (10).

Equating moments gives equation (13).

Below the bottom hinge, equation (26) applies.

The equations give:

$$\alpha = 0.72; \quad \frac{T}{\gamma H^2} = 0.0318; \quad \frac{M}{\gamma H^3} = 0.00412.$$

Failure type of the designed structures

The design is based on the type of failure which, with suitable factors of safety, leads to the most economical wall. For sands this is generally the two-hinged wall.

Brinch Hansen's hypothesis.

Brinch Hansen states that when part of an earth-retaining structure yields, the earth pressure on this part is decreased whilst the pressure against those parts which do not yield, or are forced against the earth, is increased. Therefore if a structure, designed to fail in a certain way should begin to fail in another way, this very movement will reduce the pressure on the yielding part and increase the pressure on the part designed to yield. The combined result will be that the "unforeseen" yielding ceases and that, *if the structure fails at all, it must do so in the manner assumed in its design.*

MODEL WALLS AT FAILURE

Flexibility characteristics at failure

The relation between the unit moment τ a section can withstand and its flexibility

number ρ is given by the equation⁴: $\tau = \frac{\psi}{3\sqrt{H\rho^3}}$ (38)

The exact values of ψ depend upon the allowable stress and the shape or type of section. For concrete sections, with tension and compression reinforcement of either mild or high-tensile steel, and concrete cube compression strength between 5,000 and 8,000 lb/sq. in. at 28 days, the values of ψ are approximately as follows:

Working stress.	0.25.
Steel yield	0.53.
Just prior to ultimate collapse . . .	1.30.

Similar values at yield apply to steel piling.

It is necessary that model walls at failure should have approximately the same deflected shape as the equivalent field pile in order that the earth stresses and strains

ould correspond. Therefore the flexibility numbers of model and field pile must be the same at failure at the same unit moments.

From equation (38) it follows that the following condition must be satisfied:

$$\left[\frac{\psi}{3\sqrt{H}} \right]_{\text{model}} = \left[\frac{\psi}{3\sqrt{H}} \right]_{\text{prototype}} \quad \dots \quad (39)$$

the model values of H ranged from 2.50 ft to 3.50 ft, whilst field values range from 10 ft to 60 ft.

Using mean values, at yield, the value of ψ required is given by:

$$\psi_{\text{model}} = 3\sqrt{\frac{3}{40}} \times 0.53 = 0.224 \quad \dots \quad (40)$$

for a flat plate^{1, 4}

$$\psi = \frac{2f}{E^{2/3}} \quad \dots \quad (41)$$

the flat tinned iron plate, suitable for the experiments, had a yield stress $f = 20$ tons/sq. in. and $E = 30 \times 10$ lb/sq. in., whence $\psi = 0.92$. Thus, whilst flat sections are suitable for experiments in the working stress range where the stress in the model need not equal the stress in the field pile for the same flexibility number, the flexibility is too great at yield. A study of all available alternative metals and plastics showed that decrease in yield stress f is always accompanied by a decrease in modulus E . Therefore, in order to lower the effective yield moment at a constant flexibility

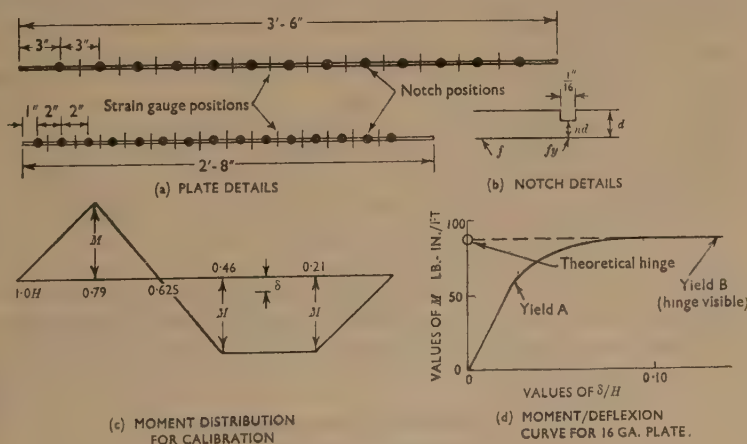


FIG. 6.—MODEL WALL DETAILS

number, a series of notches was machined in the tinned iron plate (Fig. 6a). Neglecting stress concentration at the root of the notch, the stress f near a notch when the field stress f_y occurs at a notch (Fig. 6b) is:

$$f = n^2 \cdot f_y \quad \dots \quad (42)$$

substituting f , equation (42) in equation (41) and equating (40) and (41):

$$0.224 = 2 \cdot n^2 \cdot \frac{f_y}{E^{2/3}} = n^2 \times 0.92$$

hence

$$n \approx 0.50.$$

The presence of the notches also increases the flexibility, i.e., they have an effect similar to that of reducing E . Therefore it was necessary to increase the notch depth to counteract this effect. Also the yield strength of the plate increased with decrease in plate thickness so that each section used required individual design. The final values chosen from successive flexure tests on trial sections subjected to a bending moment distribution similar to that for a flexible pile, Fig. 6c, were $n = 0.40, 0.38$, and 0.34 for 14, 16, and 18 gauge plate respectively. Flexure tests on the sections finally used gave the results shown in Fig. 7. They show that the three plates had the same flexure at yield as for the average case for field piling, with the relation

$$\tau = 0.156\rho^{-2/3} \quad \dots \dots \dots (43)$$

The shape of the moment/deflexion curve shown in Fig. 6d is similar to that for reinforced concrete and corrugated-type steel sections in bending and has an important influence on the behaviour of walls at failure. The yield point A was taken as that point at which a definite divergence from the almost linear initial relation could be detected on plotting the results, whereas the ultimate yield B was that

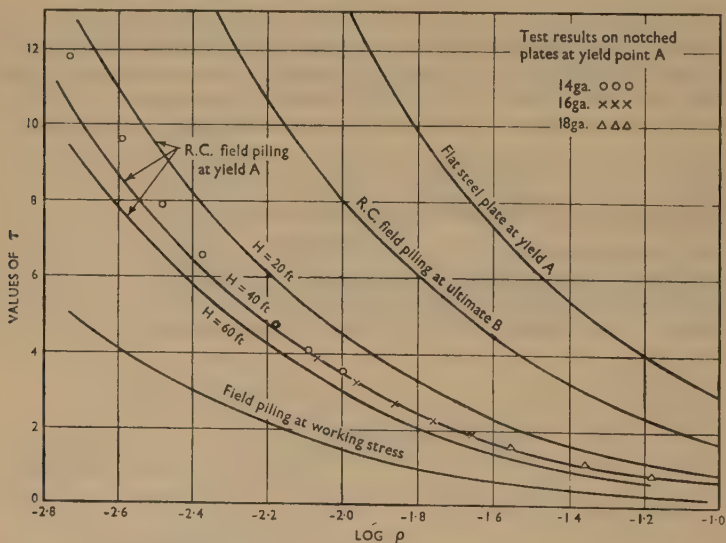


FIG. 7.—FLEXIBILITY CHARACTERISTICS

associated with a visible “yield hinge” where an angle from the straight of about 10° or greater formed at a notch.

The walls could be straightened after test and used again with very little change in the flexure and yield properties which are given in Table 2. The values of yield B in any given test varied a little from the average values shown. If the final hinge formed slowly the stage at which it was clearly visible was recorded. Often it formed rapidly when a larger permanent deformation and yield moment occurred. The flexibility numbers may be taken as constant up to yield A and thereafter increase by an amount of the order of $\log \rho = +0.5$. All the results are given in terms of the flexibility numbers up to first yield.

TABLE 2

	log ρ			$\tau = \frac{M}{H^3}$ lb.-in./ft ⁴ units					
H: in.	14	16	18	Yield A			Yield B		
				14	16	18	14	16	18
28	-2.72	-2.20	-1.90	11.80	4.72	2.91	19.7	8.2	5.5
30	-2.59	-2.07	-1.77	9.64	3.84	2.37	16.0	6.7	4.5
32	-2.48	-1.96	-1.66	7.90	3.16	1.95	13.2	5.5	3.7
34	-2.37	-1.86	-1.56	6.61	2.64	1.63	11.0	4.6	3.1
36	-2.28	-1.76	-1.46	5.56	2.23	1.37	9.3	3.9	2.6
38	-2.18	-1.66	-1.36	4.74	1.89	1.17	7.7	3.3	2.2
40	-2.09	-1.57	-1.27	4.05	1.62	1.00	6.7	2.8	1.9
42	-2.00	-1.49	-1.19	3.51	1.40	0.87	5.8	2.4	1.6

DESCRIPTION OF APPARATUS

The apparatus is shown in Fig. 8, p. 290, and Fig. 9, p. 296. The walls were 7 ft 2 in. wide consisting of seventeen sections 5 in. wide. The sections were held at the top and bottom by means of 14 gauge plate, 1 in. wide A, and were joined down their sides by cellulose tape on both sides of the wall B. This simulated the individually driven piles and prevented moment decrease arising from anticlastic curvature associated with wide plates. Strain gauges were mounted midway between the notches on both sides of the wall on the centre section. The wiring was arranged so allow complete freedom for excessive bending at any notch. Yield at the notches did not influence the stress at the gauges which operated under a working stress up to and including ultimate collapse. Thus a linear calibrated relation between applied bending moment and strain gauge reading occurred throughout the tests.

During setting-up, the wall was suspended from an angle C, which spanned the width of the bin. Two such angles were used to clamp the wall for the *encastré* tests. Two tie-rods D $\frac{1}{4}$ in. in diameter, anchored the wall to the rear concrete wall of the bin. The tie-bars were machined to a flat section towards the rear, the section being 0.060 in. thick on one pair of tie-bars and 0.030 in. thick on an alternate pair. Strain gauges mounted at these sections were used to record the tie-rod load, the alternate tie-bars being available to maintain high sensitivity at the various wall scales. Since the tie-rod spacing was necessarily larger at model scale than normal in the field, a heavy angle E, was used for a waling to reduce flexure, and therefore differential yield along the wall to within a few thousandths of an inch. The weight of this waling, large relative to that of the wall, was counterbalanced by means of two lead weights F. Vertical and horizontal tie yields were measured by dial gauges reading to 0.001 in. The outward wall deflexions above the dredge level, which were quite large, were measured by means of a steel rule to 0.10 cm at three sections against plumb lines. Settlement of the backfill was measured by placing a row of wooden footings G on the sand surface, each of which carried a light rod. The rods passed through wide clearance holes in an angle H, readings being taken between the top of the rods and the angle by steel rule to 0.10 cm.

The wall at the anchor level could be prevented from vertical slip by means of a

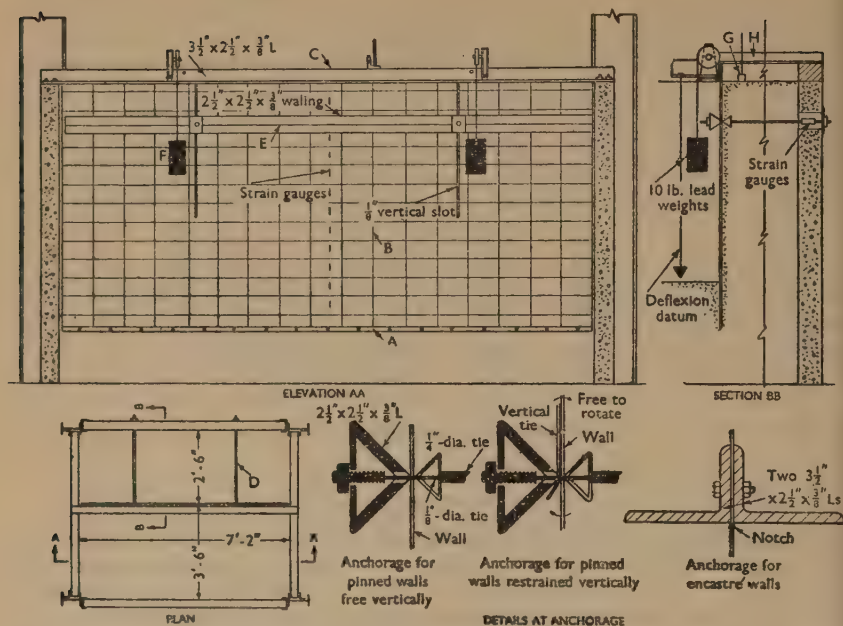


FIG. 8.—ARRANGEMENT OF APPARATUS

pair of thin steel strips J, connecting the tie-rods to an overhead girder. The wall remained perfectly free to rotate about the anchor point.

TYPES OF EXPERIMENT

The number of variables influencing the performance of model piles embedded in dry cohesionless soil are too large to investigate systematically in one Paper. Nevertheless an understanding of the existence of these variables, which may occur naturally in a large number of combinations, is of considerable importance, since it will be found that any one theory or design procedure is valid only under certain conditions. The experiments were made using combinations of the following conditions.

Method of construction

- (a) *Driven-and-dredged walls*:—The bin was filled on both sides of the wall up to the anchorage. The wall was released, the tie-rods set to be just tight, and all zero readings were taken. Backfilling was completed and the sand dredged from the outside.
- (b) *Backfilled-and-dredged walls*:—The bin was filled to $\alpha = 0.7$ when the wall was released and zero readings taken. The wall was then backfilled, the sand in front being subsequently dredged until failure.

Subsidence

- (a) *Subsidence allowed*:—The readings were recorded after the backfill level had dropped, assuming that this was “failure” and that in practice the level would not be made up.
- (b) *Subsidence made up*:—The backfill level was kept constant during failure in order to compare the observed readings with the theory.

Anchor yield for pinned walls

- (a) *Small natural yield*:—The yield increased gradually between $H/2,000$ and $H/1,000$ throughout the test as a result of elastic deformation of the whole anchorage system.
- (b) *Medium natural yield*:—Rubber washers were inserted at the rear ends of the tie-rods to give yields up to $H/500$.
- (c) *Large natural yield*:—Yield up to $H/200$.

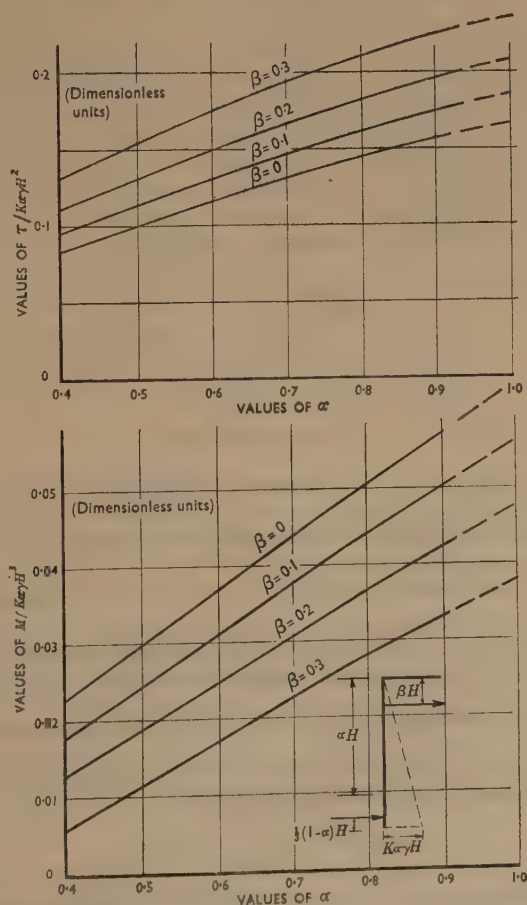


FIG. 10.—FREE-EARTH-SUPPORT VALUES FOR UNIFORM COHESIONLESS SOIL

- (d) *Artificial yield*:—Yield additional to (a), (b), or (c) could be imposed by release of the tie-rod nuts at the waling.
- (e) *No vertical yield*:—Vertical anchor slip was prevented by vertical tie-bars.

Anchorage encastré.

Anchor level. Practical range $\beta = 0$ to 0.25.

Surcharge.

Wall scale and section. (See Table 2).

Subsoil and/or backfill type.

Dense sand, loose sand and loose silt were used with properties previously recorded.²

THE FREE-EARTH-SUPPORT CALCULATION

Change in the anchor level between tests, or in the dredged level during a test, results in changes in the anchor load and maximum bending moment arising from two types of cause. The first is directly due to the variation in the geometry of the wall span, taken into account by the free-earth-support (F.E.S.) calculation. The second results from varying degrees of penetration fixity and passive pressure above the tie-rods. It is required to study the influence of the variables of the problem on the moments and anchor loads from this second cause alone; this may be achieved by expressing the results as ratios of the F.E.S. values.

For the case of level soil surface with no surcharge, and given penetration coefficient α , the following dimensionless coefficients are obtained¹:

$$\frac{T}{K_a \gamma H^3} = \frac{0.5\alpha}{2 + \alpha - 3\beta} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (44)$$

$$\text{and} \quad \frac{M_{\max}}{K_a \gamma H^3} = \frac{0.5\alpha}{2 + \alpha - 3\beta} \left[2 \left(\frac{\alpha}{2 + \alpha - 3\beta} \right)^{1/2} - \beta \right] \quad \cdot \quad \cdot \quad (45)$$

For loose sand $\phi = 30^\circ$; $K_a = 0.266$; $\gamma = 95$ lb/cu. ft

so that $K_a \gamma = 25$ lb/cu. ft

$$\text{and} \quad \tau = 12 \times \frac{M_{\max}}{H^3} \frac{\text{lb. in.}}{\text{ft}^4} \text{ units.}$$

Values calculated from equations (44) and (45) are plotted in Figs 10.

BRINCH HANSEN'S THEORY AND THE FLEXIBILITY NUMBER

The flexibility number does not enter into Brinch Hansen's theory, but in order to plot the theoretical results at failure on the diagrams of test results, it is necessary to know the flexibility number to which each particular design corresponds.

The values of the coefficients α , $\frac{M}{\gamma H^3}$, and $\frac{T}{\gamma H^2}$ calculated for the various conditions at failure and by the F.E.S. calculation at corresponding values of α are shown in Table 3. The values of τ at failure are given by:

$$\tau = 1,140 \times \left(\frac{M}{\gamma H^3} \right) \text{ lb. in./ft}^4$$

and equating this value to τ in equation (43) the corresponding values of $\log \rho$ at yield A are obtained. These values, together with the moments and tie loads

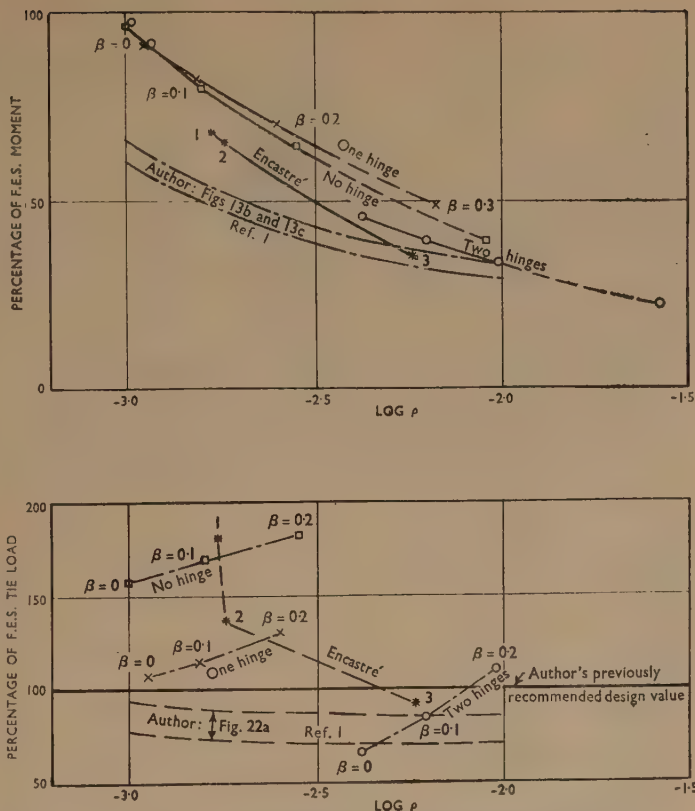


FIG. 11.—VARIATION OF FLEXIBILITY NUMBER AT THEORETICAL FAILURE

expressed as a percentage of the F.E.S. calculation are included in Table 3 and have been used to plot Fig. 11 and the position of results from the failure theory on subsequent diagrams. As would be expected the flexibility numbers increase with failure state change from Figs 1a to c since the pile length increases while the section size decreases.

DESCRIPTION OF RESULTS

All the tests recorded in the Paper were made using dry loose sand. Examples of the data obtained directly from each test are given in Figs 12–14 for the particular case of $\beta = 0.2$, driven-and-dredged construction and naturally yielding anchors, one test including an artificial tie-rod release. The results of all other tests, which were first plotted as in Figs 12–14 are collected in Figs 16–30. It is convenient to describe the results under the following headings.

Backfill subsidence

The maximum wall deflexions, δ (Fig. 12a), increased with dredging in a curve similar in shape to the moment/deflexion curve in the bending test (Fig. 6d). When the yield point A was reached there was no visible evidence of wall failure or yield.

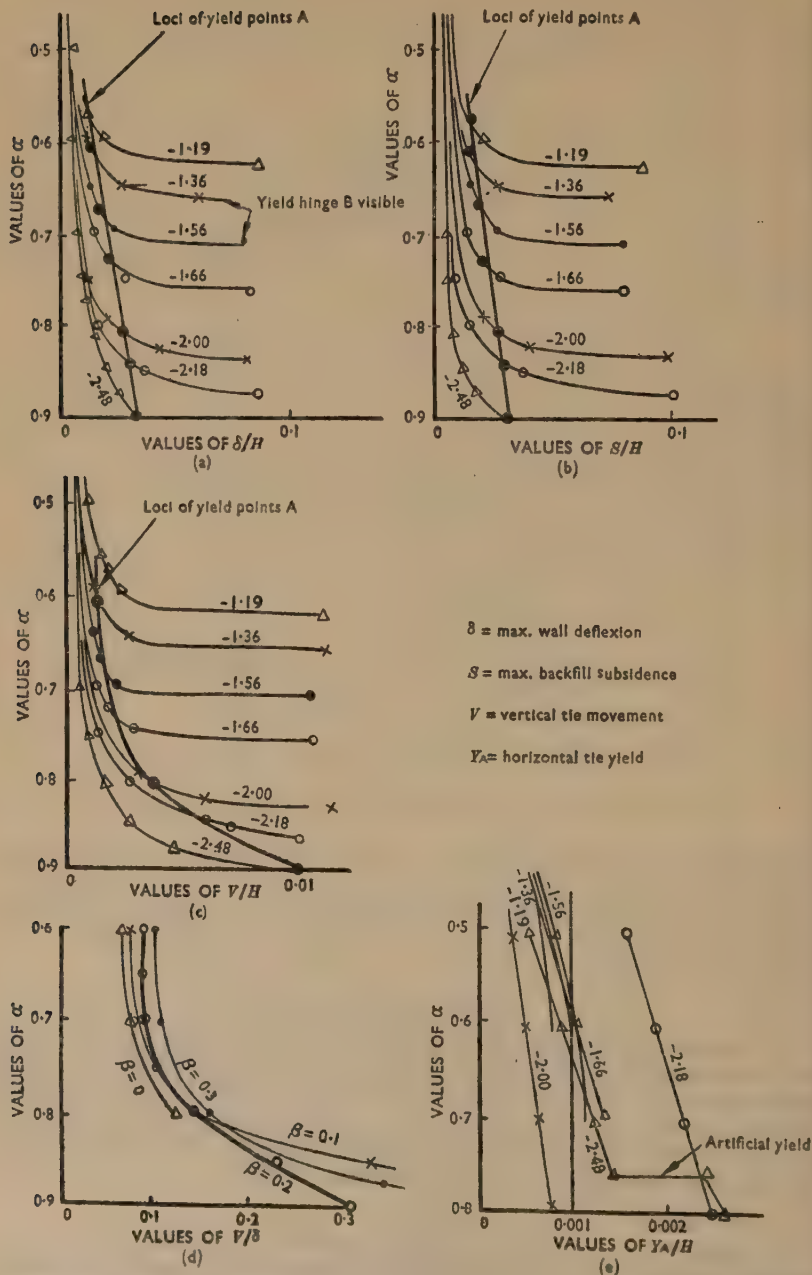
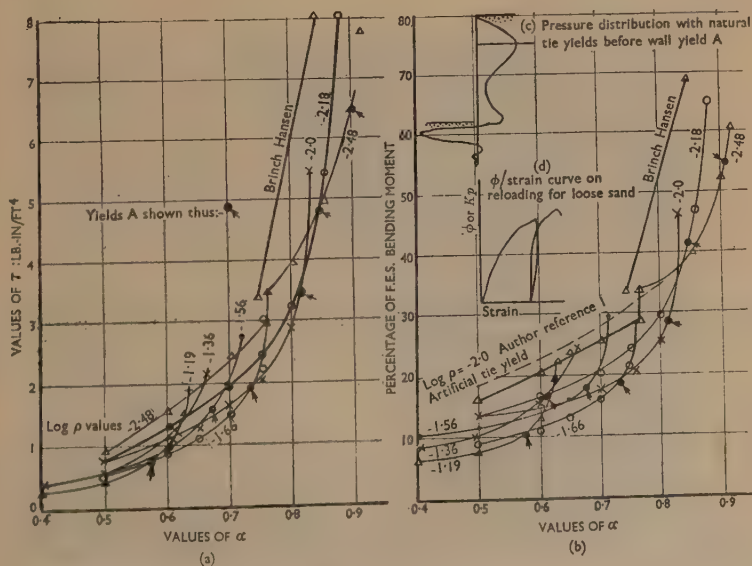


FIG. 12.—WALL DEFLEXIONS AND BACKFILL SUBSIDENCE WITH DREDGING. $\beta = 0.2$

TABLE 3

No. of Hinges	β	α	Failure theory		F.E.S. calculation		% of F.E.S.		$\log p$
			$\frac{M}{\gamma H^3}$	$\frac{T}{\gamma H^2}$	$\frac{M}{\gamma H^3}$	$\frac{T}{\gamma H^2}$	M	T	
Pinned walls	0	0	0.820	0.0131	0.062	0.0136	0.0386	96 159	-2.98
		0.1	0.830	0.0097	0.075	0.0122	0.0440	80 169	-2.78
		0.2	0.845	0.0067	0.091	0.0104	0.0500	65 181	-2.54
		0.3	0.870	0.0031	0.115	0.0081	0.0585	38 197	-2.04
	1	0	0.815	0.0122	0.042	0.0135	0.0386	90 107	-2.93
		0.1	0.822	0.0099	0.050	0.0122	0.0436	81 114	-2.79
		0.2	0.837	0.0071	0.066	0.0101	0.0500	70 132	-2.57
		0.3	0.855	0.0038	0.093	0.0080	0.0580	47 160	-2.17
	2	0	0.698	0.0052	0.023	0.0114	0.034	45 69	-2.36
		0.1	0.715	0.0040	0.033	0.0103	0.039	39 84	-2.20
		0.2	0.740	0.0030	0.051	0.0086	0.046	35 110	-2.01
		0.3	0.797	0.0015	0.086	0.0072	0.055	21 155	-1.56
Encasement	1		0.830	0.0094	0.070	0.0138	0.039	68 180	-2.75
	2		0.826	0.0090	0.053	0.0137	0.039	65 138	-2.73
	3		0.720	0.0041	0.032	0.0117	0.035	35 91	-2.22


 FIG. 13.—VARIATION OF MAXIMUM BENDING MOMENT WITH DREDGING. $\beta = 0.2$

The yield hinge or hinges appeared only when large deflexions occurred at yield point B (Figs 15d and f). The backfill subsidence S (Fig. 12b) varied with dredge level and wall flexibility in an identical manner. (Compare Figs 12a and b.) Thus at yield A subsidence was of the order of $1/50 \times$ the wall height and becoming visible (Fig. 15a). Immediately following A a passive failure occurred above the tie as the sand subsided (Fig. 15b), followed by an active slip outcrop farther from the wall (Fig. 15c). The subsidence increased in size at the formation of yield B (Figs 15d and 15e) and became catastrophic as the wall rotated to the final state (Fig. 15f). This had an important influence on the wall stress at failure. In the majority of the tests the backfill was made up to the theoretical level before dredging further, to

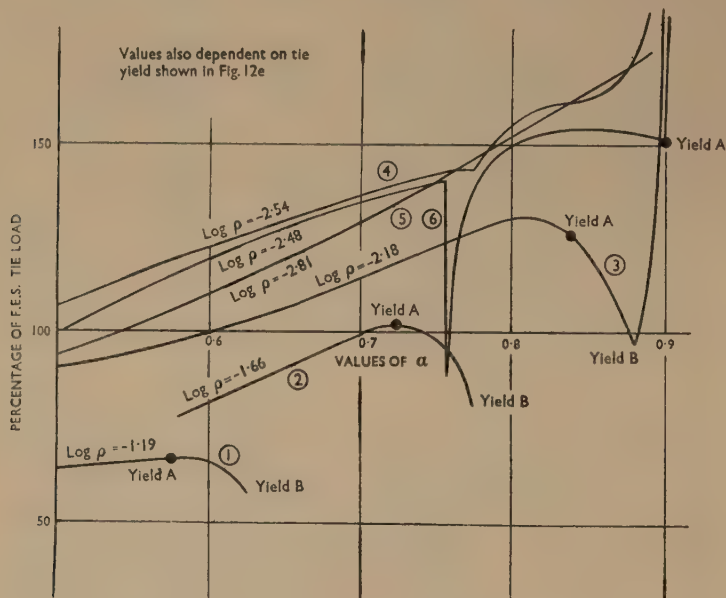


FIG. 14.—VARIATION OF TIE LOAD WITH DREDGING. $\beta = 0.2$

eliminate the influence of a varying height of backfill. After each dredge in the region between yields A and B, readings were taken before the backfill was made up.

Vertical wall slip

The vertical deflexions V of the anchorage are plotted against α in Fig. 12c. The deflexions result partly from the wall bending and partly from settlement of the toe of the pile. The vertical slip from bending should be approximately proportional to the maximum wall deflexion δ . Fig. 12d shows that the ratio V/δ commences to increase rapidly beyond $\alpha = 0.75$, independent of the value of β . This is attributable to settlement and finally failure of the toe of the wall under the vertical thrust of the backfill.



FIG. 9.—PINNED WALL AT YIELD B
($\log \rho = -1.36$; $H = 38$ in.; $\beta = 0.2$; $\alpha = 0.66$)



FIG. 15a.—WALL AND BACKFILL DEFORMATIONS BETWEEN YIELDS A AND B
(Yield A; $\alpha = 0.74$; $\beta = 0.2$; $\log \rho = 1.76$; subsidence $1/40$ at wall height)



FIG. 15b.—WALL AND BACKFILL DEFORMATIONS BETWEEN YIELDS
A AND B ($\alpha = 0.75$)



FIG. 15c.—WALL AND BACKFILL DEFORMATIONS
($\alpha = 0.76$)



FIG. 15d.—YIELD B COMMENCES ($\alpha = 0.78$)



FIG. 15e.—SUBSIDENCE ONE-TENTH OF WALL HEIGHT ($\alpha = 0.78$)

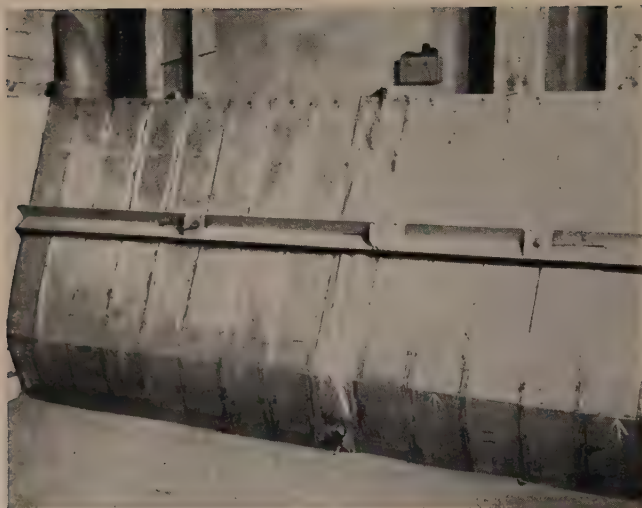


FIG. 15f.—YIELD HINGE FORMS. SUBSIDENCE LARGE ($\alpha = 0.79$)



FIG. 29.—ENCASTRÉ WALL BEYOND YIELD B WITH PASSIVE SLIP
($\log \rho = -1.62$; $H = 39$ in.; $\alpha = 0.85$)

Wall friction

Vertical wall slip did not show any evidence, in tie loads or bending moments, of causing a temporary decrease in wall friction below the tie level. This was because the wall moved under thrust from the backfill which, owing to outward wall deflexion, was always moving downward relative to the wall. (Compare Figs 12b and c.) Had the wall been forced downwards by an external force relative to the backfill, wall friction would have been reduced. Negative wall friction above the tie arising from rotation of the wall into the backfill was probably reduced by backfill subsidence relative to the wall. Prevention of vertical slip by means of vertical ties gave a vertical tie load equal to the theoretical value taking the angle of wall friction $= \phi$. This load remained practically constant throughout the test.

Maximum penetration coefficient α

Passive failure, by outward sliding of the toe, rarely occurred when the anchorage was below the top of the wall (Fig. 16). As the region of passive failure approached

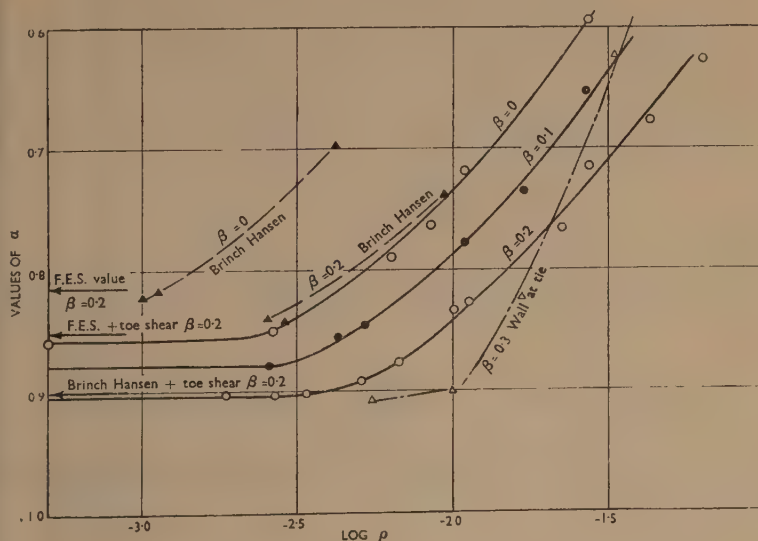


FIG. 16.—LOCUS OF FAILURE AT YIELD POINTS B. VERTICAL AND PASSIVE SLIP

$= 0.85$ to 0.92 , the vertical toe slips increased to a rate which equalled that of dredging. Thus, toe-base failure preceded that of passive slip and maintained a minimum penetration depth. At the limit, small passive slips and large vertical slips occurred simultaneously. The passive slips shown in Fig. 9 (reference 1) applied to a wall of greater bearing capacity, where toe-base failure did not occur.

Separate experiments are required to determine the relation between the bearing capacity at the toe of the model and that at the toe of a field pile. However, in the light of this result, it is of interest to read the discussion⁵ on toe-shear force. Very high vertical pressures do occur with steel sheet-piling which is therefore more

likely to fail in bearing on a loose subsoil than by passive slip. If passive slip occurs at all, toe shear will act. The ultimate values of α (Fig. 16) are predicted theoretically only by including toe shear in Brinch Hansen's values. Where backfill subsidence decreases the passive resistance above the ties, it also decreases the whole active moment, so that subsidence does not influence the ultimate value of α .

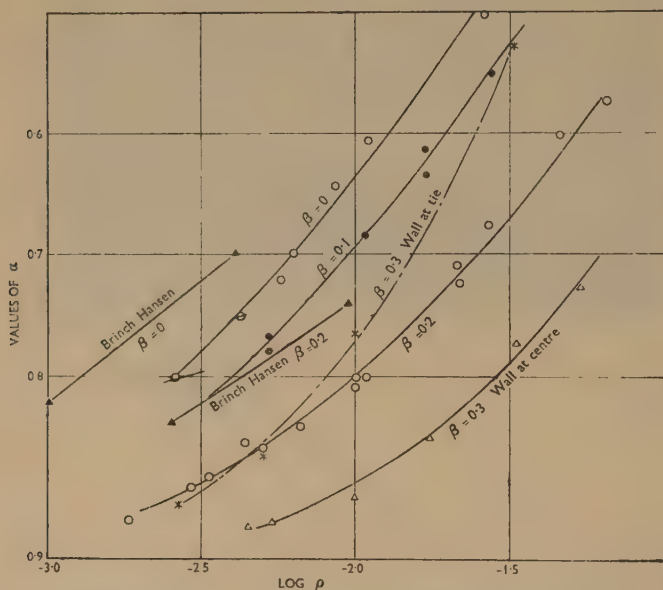


FIG. 17.—LOCI OF YIELD POINT A WITH DREDGING

The values of α at yield point A (Fig. 17) enable the location of this yield point on subsequent diagrams.

Bending moments

The variations of bending moments with dredging for tests with $\beta = 0.2$ are given in Fig. 13 and moment/flexibility curves are given in Figs 18–21. Figs 13 and 20 show that before yield A was reached for tests with $\beta = 0.2$, the moments were reduced below those previously published for the case of artificially released anchorage. No such further reduction occurred for the case of $\beta = 0$ and very little for $\beta = 0.1$ (Figs 18 and 19). The reduction arose from passive pressure above the ties as the wall rotated about the anchor point and may have involved slight arching between the anchorage and the dredge level (Fig. 13c). As yield A was reached, however, the rate of moment increase with dredging accelerated (Fig. 13b), rising to a maximum value at yield B to exactly those values previously published for the case of a triangular active pressure distribution. The rise in moment at yield A is shown more clearly in Fig. 20. It was directly due to backfill subsidence into the

mid left by the deflecting wall, thus reducing the passive pressure above the ties and causing a triangular active pressure distribution. When yield A occurred beyond $\alpha = 0.8$ with stiffer walls fixity breakdown also caused an increase in moment. The volume increases behind the wall were therefore larger and the subsidence at yield A increased with α as shown in Fig. 12b. Subsidence after yield A was then more

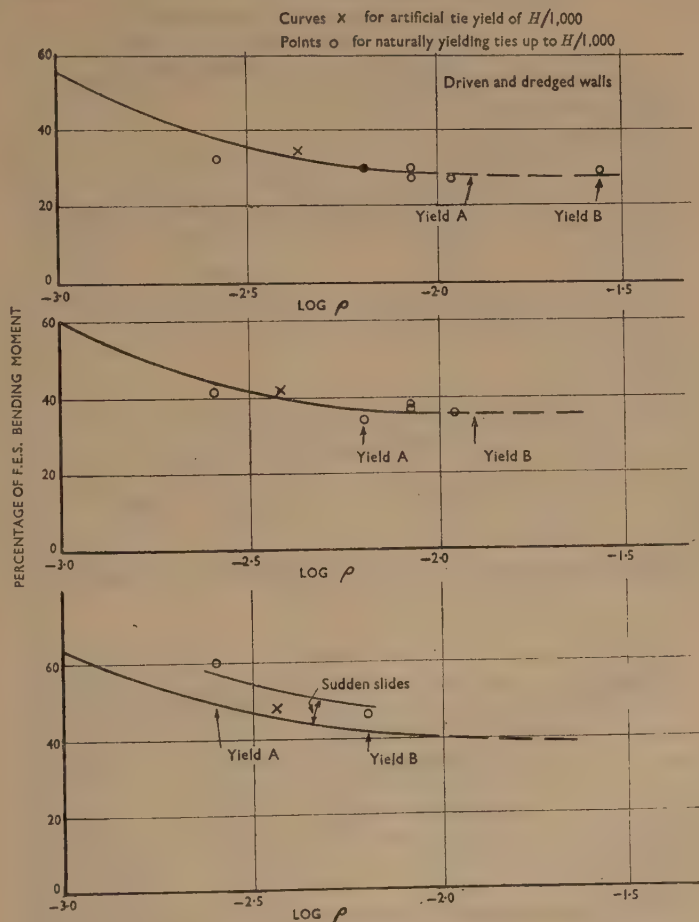


FIG. 18.—MOMENT/FLEXIBILITY CURVES FOR $\beta = 0$

violent so that the final moments exceeded the values obtained under static conditions (Fig. 19).

Tie-rod loads

The tie yield Y_A depended on the tie load, which varied as the square of the scale. Therefore it was difficult to run a series of tests with naturally yielding tie-rods with

identical ratios Y_A/H at each stage. The curves shown in Figs 23–25 were obtained by interpolation from points of varying yield between $0.0001H$ and $0.004H$. They show that the maximum tie load for a given yield occurred in the working stress region and commenced to decrease before wall yield A was reached. At yield B the load decreased to the minimum values irrespective of the degree of tie yield. The

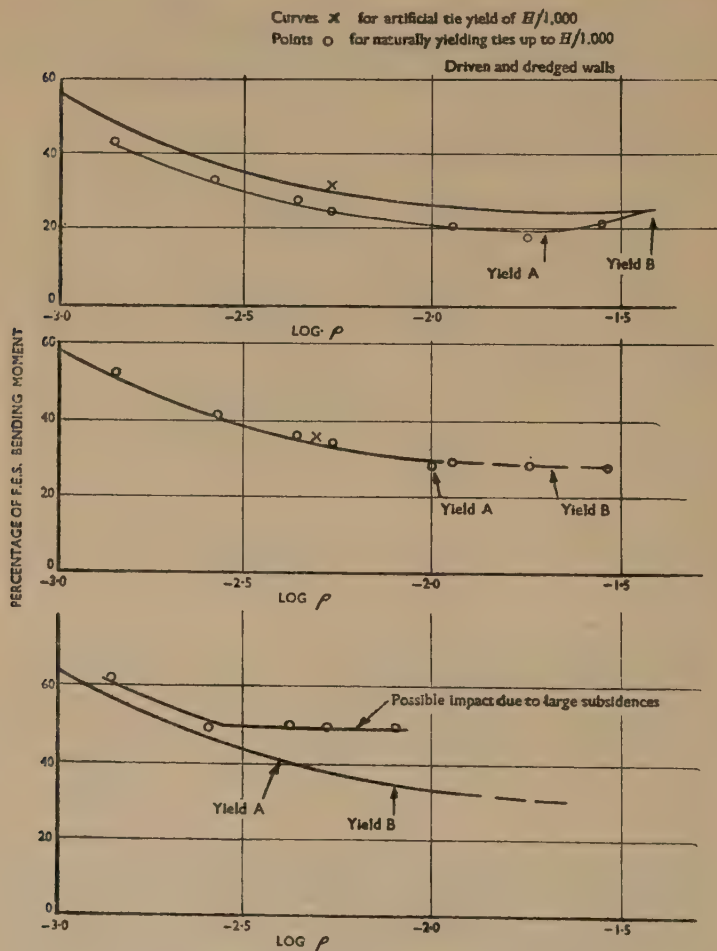


FIG. 19.—MOMENT/FLEXIBILITY CURVES FOR $\beta = 0.1$

variations of tie load at wall yield A with α and β irrespective of wall flexibility, are given in Fig. 27, and with a natural tie yield of $H/1,000$ (Fig. 27d), the load equalled the F.E.S. value at $\alpha \approx 0.7$. The maximum values in the working stress region exceeded the F.E.S. values by a factor varying between 1.0 and 1.25 in the practical design range (Fig. 27e).

Subsidence and vertical wall slip influenced the tie load in opposite ways, illustrated in Fig. 14. Curves 1 and 2 applied to piles which failed in bending before toe slip commenced, when the tie load decreased to the minimum at yield B and remained there. Curve 3 applied to a stiffer wall which reached yield A as vertical slips were increasing. After the tie load decreased to a minimum at yield B a slight additional dredge caused a large vertical toe slip which immediately resulted in a large increase

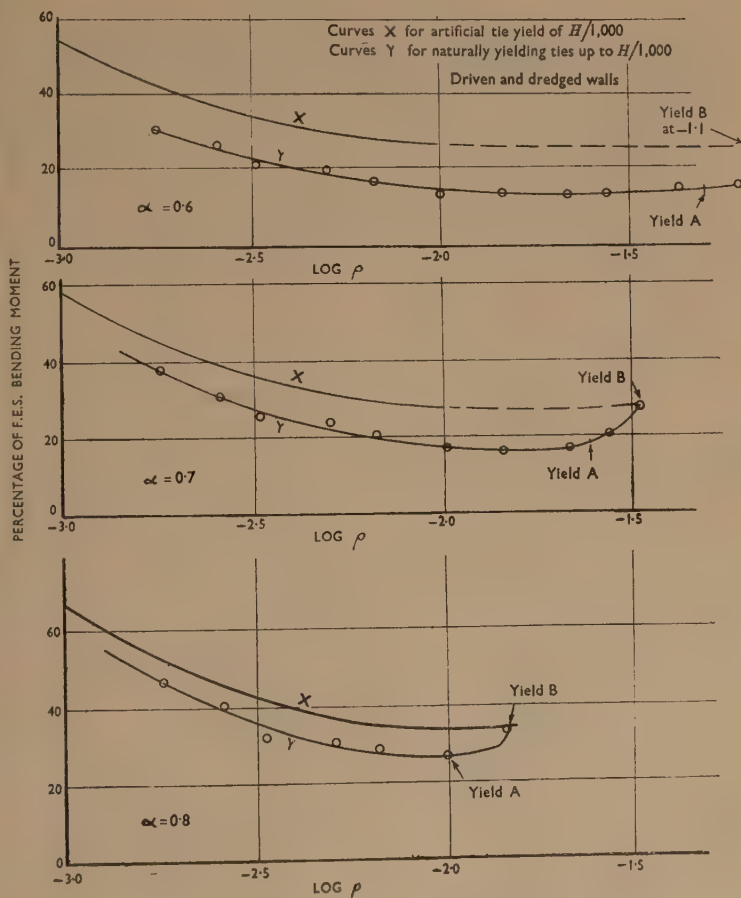


FIG. 20.—MOMENT/FLEXIBILITY CURVES FOR $\beta = 0.2$

in tie load. Small outward passive slips also occurred with the large toe slips. However, when the toe slip was prevented by means of vertical ties, passive slip did not increase the tie load as expected from Brinch Hansen's "no-hinge" failure, since subsidence acted with the outward wall rotation. Large vertical wall slip therefore

counteracted subsidence and presumably maintained wall friction above the tie level. It is also possible that some of the vertical thrust was transferred directly to the tie-rods after slip. Curves 4 and 5 for stiffer walls in the working range which did not reach yield A showed no load decrease. In the region $\alpha > 0.85$ subsidence exceeded wall slip away from the wall but both movements were small enough to give a final "no-hinge" failure state.

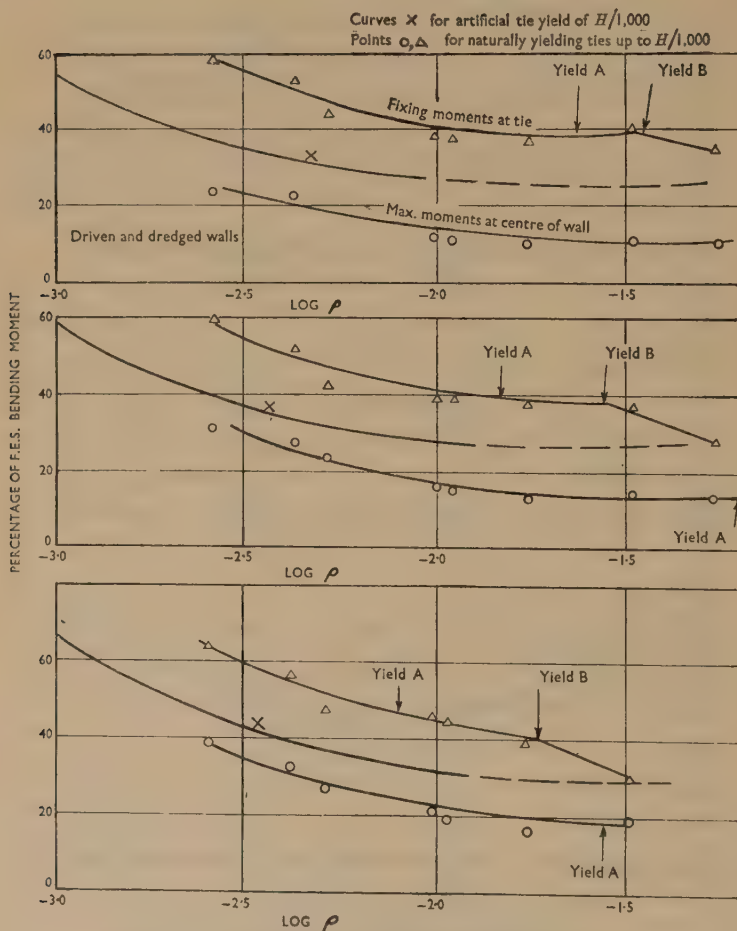
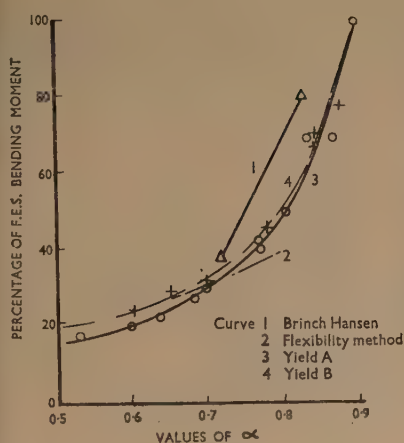
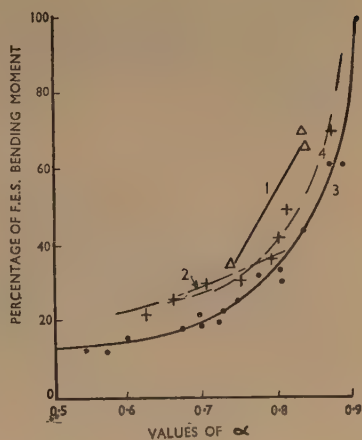
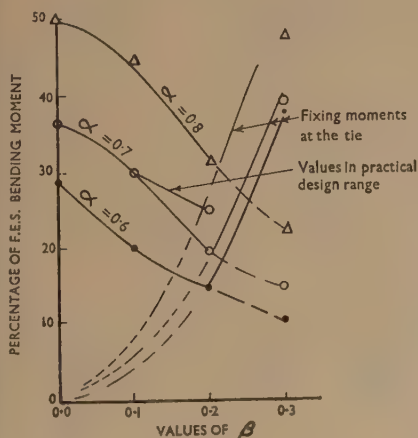
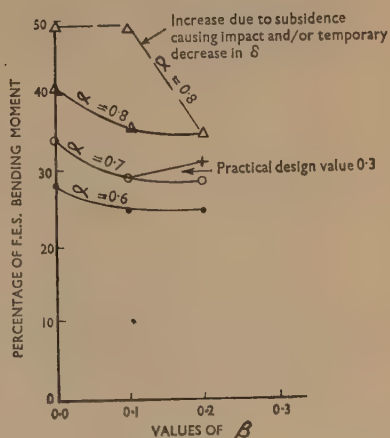


FIG. 21.—MOMENT/FLEXIBILITY CURVES FOR $\beta = 0.3$

Artificial release of the tie-rods caused an immediate decrease in load (curve 6) whilst subsequent dredging caused the load immediately to increase to its previous value. This is connected with the shape of the unloading and reloading ϕ /strain curve for sands (Fig. 13d).


 (a) $\beta = 0.1$

 (b) $\beta = 0.2$


(c) Yield A



(d) Yield B

 FIG. 22.—VARIATIONS OF BENDING MOMENT WITH α AND β AT YIELD

Encastred walls

The results shown in Fig. 28 indicate that, in contrast to pinned walls, the maximum bending moments decreased towards failure. This was due to the following reasons.

- (1) No moment reduction from passive pressure above the tie could act before yield, thus eliminating this source of moment increase on subsidence.
- (2) Subsidence was allowed to continue without making up to level, as under field conditions with level backfill.
- (3) The vertical thrust reaction at the anchorage gave a restoring moment at the point of maximum deflexion equal to the product of the force and the deflexion.

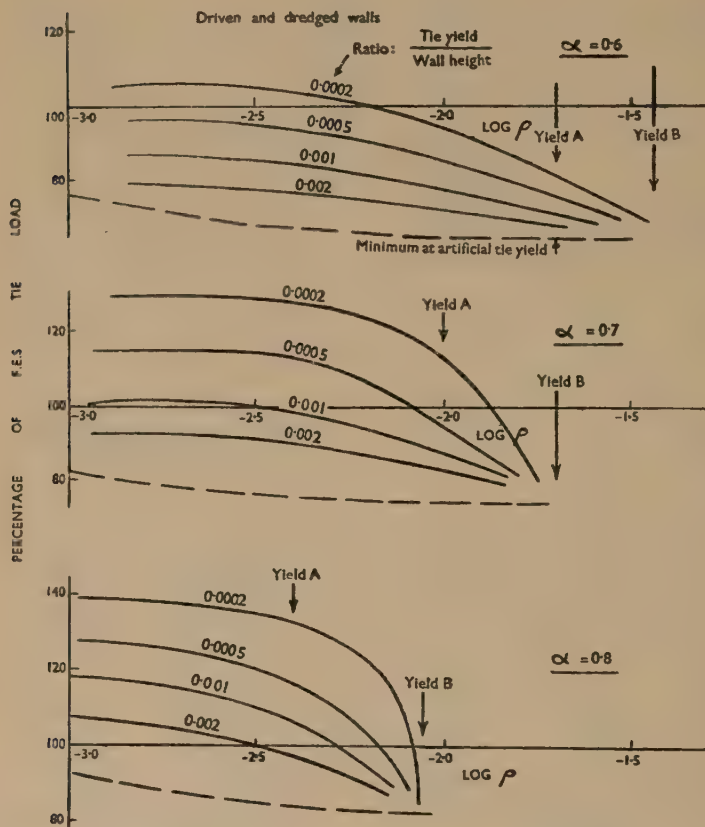


FIG. 23.—TIE-LOAD/FLEXIBILITY CURVES FOR $\beta = 0.1$

Owing to cause (3), no single yield hinge B was ever visible in the same way as for pinned walls (compare Fig. 29, facing p. 297, with Figs 9 and 15f). In the test illustrated in Fig. 29, the backfill was made up in an attempt to induce sudden failure, but subsidence with passive slip continually lowered the backfill level at a rate equal to that of backfill, whilst the thrust maintained a continuous wall curvature.

For these reasons, catastrophic wall failure could not be induced to occur.

Economic wall section

The cost of the material in the wall is proportional to the volume, thus:

$$\text{Cost} \propto H \cdot d$$

where d denotes the wall thickness. The maximum moment M is related to the wall thickness by the relation:

$$M = K \cdot d^2$$

where K denotes a strength and shape factor.

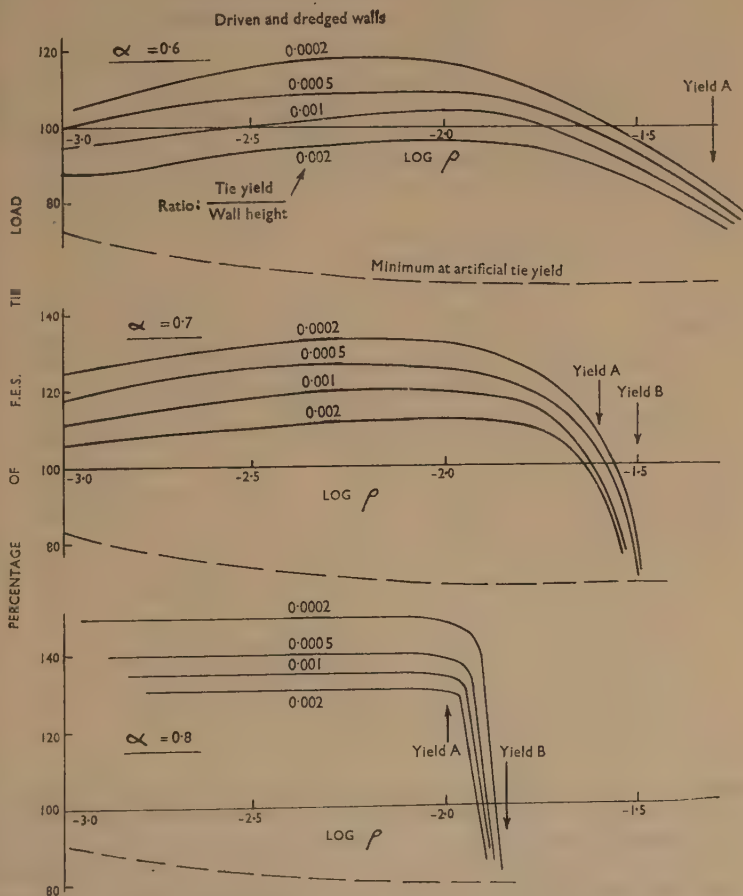


FIG. 24.—TIE-LOAD/FLEXIBILITY CURVES FOR $\beta = 0.2$

Eliminating d from the above equations:

$$\text{Cost} \propto H \cdot \sqrt{M}$$

Writing the free height to be retained, αH equal to h , and substituting τH^3 for M :

$$\text{Cost} \propto \sqrt{\tau \left(\frac{h}{a}\right)^5}$$

For a given free height h required, the cost is proportional to $\sqrt{\tau/\alpha^5}$, which will be called the cost index.

Experimental values of the cost index at all yield points A are plotted against α for natural tie yields $\leq H/1,000$ in Fig. 30. The graphs show that the minimum cost index occurs over a range of α values, an average value for all values of β being 0.73. For $\alpha < 0.70$ the cost increases with the greater pile length. For $\alpha > 0.75$ the cost increases with greater wall thickness required with increasing moments. Plotting

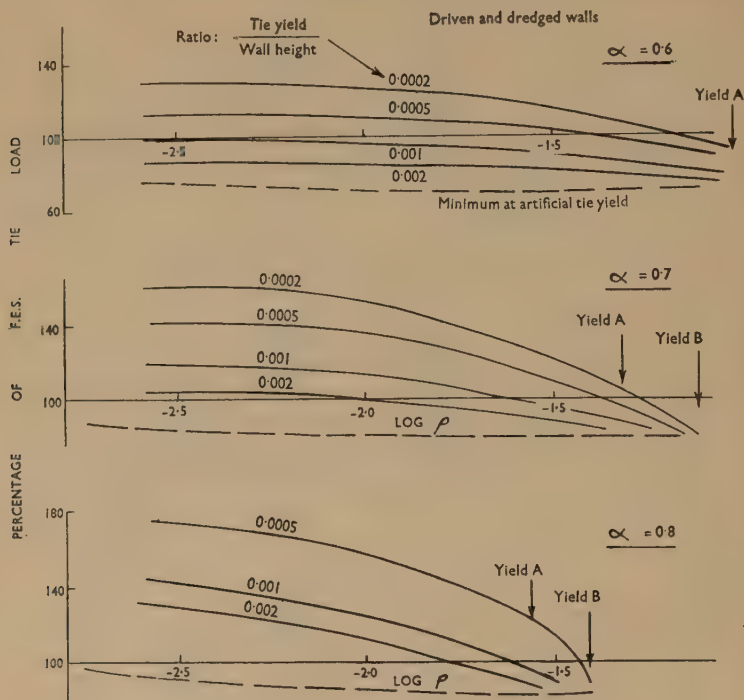


FIG. 25.—TIE-LOAD/FLEXIBILITY CURVES FOR $\beta = 0.3$

the minimum values of the index against β with allowance for the increased moments at the tie beyond $\beta = 0.2$, the dimensions of the economical wall design in loose sand are obtained (Figs 30b and c). These agree precisely with the two-hinge failure by Brinch Hansen, Table 3, which he stated to be the most economical for loose sand. The wall shown in Fig. 15a has these dimensions. The wall thickness or cost was 20% higher by Brinch Hansen than that observed.

It is clear from Fig. 30a that the exact design penetration depth has a smaller ultimate influence on wall cost than had been supposed, especially compared with the influence of tie-rod depth.

Backfilled walls

On completion of backfilling, the bending moments and tie loads were precisely equal to those given by a triangular active pressure distribution coupled with

flexibility decrease. Subsequent dredging, which might be required if an existing harbour was to be deepened, or which might simulate wall creep, led to tie load increase with passive resistance above the tie, but to much smaller values than with driven and dredged walls. This is due to the lower slope of the ϕ /strain curve on first loading compared with reloading. The influence on the bending moments was very small.

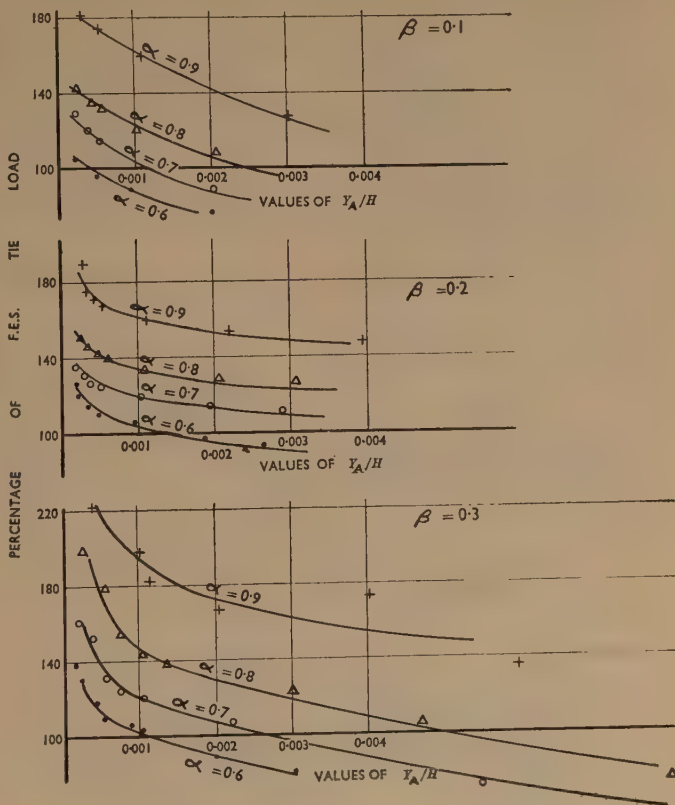


FIG. 26.—VARIATION OF MAXIMUM TIE-LOAD WITH NATURAL TIE YIELD IN THE WORKING-STRESS RANGE

Subsidence allowed

When the surface subsidence was allowed to continue without backfilling to the theoretical retained height, the ultimate bending moments and tie loads at yield B were lower than those recorded.

DISCUSSION

It is clear from the results that Brinch Hansen's theory calculates the highest bending moment, tie load, and geometry of the economical wall section very closely for loose sand and provides a sound method of design. The following discussion is intended to clarify the principles of both this method and the flexibility method in

the light of the test results obtained so far, and to study the extent to which either method is likely to be of general application.

Brinch Hansen's use of definite yield hinges as illustrated in Fig. 1a, and his reference to the ultimate strength of structural materials, suggests that his method was originally intended to apply at yield B. For concrete the ultimate compression

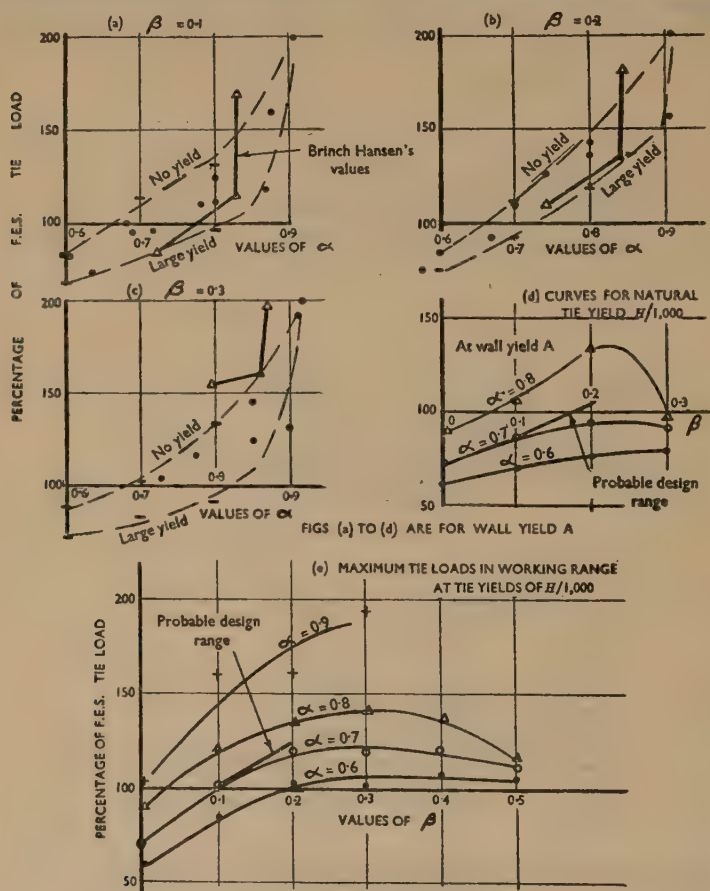


FIG. 27.—VARIATION OF TIE LOAD WITH α AND β FOR YIELD $H/1,000$

strength is used, but for steel the first yield stress is used, presumably as a result of the large strain at this point. In bending, only the outer fibres are subject to the yield stress at yield A, so that a yield hinge will not form immediately after this point. Nevertheless, if subsidence is considered to be failure, and if a calculation using passive pressure above the tie-rods is to apply at all, then ultimate strength of the wall must be taken to be the first yield point. For reinforced concrete, this point may occur before yield of the reinforcing, especially if the limit is set by the size of tension crack at which corrosion would proceed. Another limiting factor might be

the degree of settlement of surcharge, or the fatigue limit of walls subject to wave action. For these reasons, yield A will be taken as the limit by any design method in the following discussion, with the understanding that this limiting deformation may have to be reduced under special circumstances.

Comparing first Brinch Hansen's values with those observed, a good estimate of the tie load at yield A is obtained for ties yielding naturally by $H/1,000$, although the

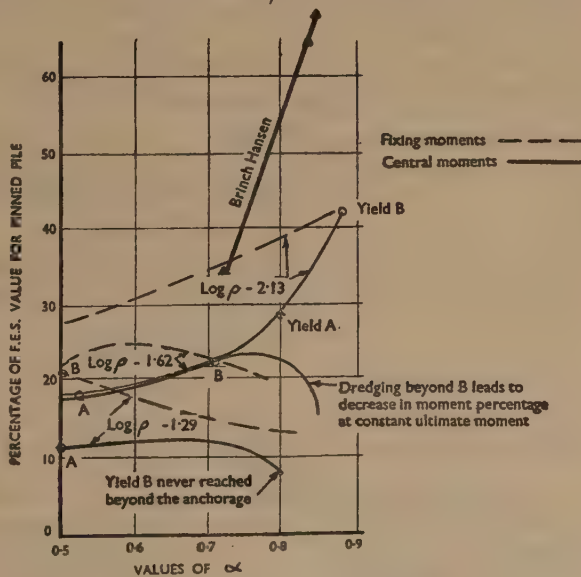
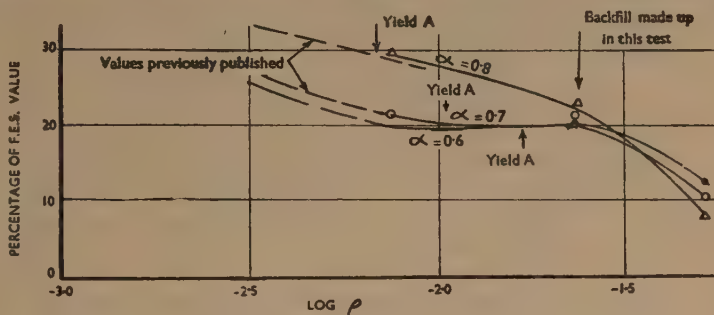


FIG. 28.—ENCASTRÉ WALL RESULTS

values are lower than those for the case of no tie yield to which the theory strictly applies. The values are lower still than those at working wall stress. For the optimum wall dimensions ($\alpha = 0.74$; $\beta = 0.2$), the values are compared in Table 4. The decrease in tie load towards yield A may be associated with the mechanism change from one hinge to two hinges, but since no hinges actually occur in this range, and subsidence is a continual process, it is difficult to separate these two effects.

On the other hand the method overestimates the bending moment at yield A for

TABLE 4

Tie yield	Wall stress	Factor on F.E.S. tie load:	
		Observed	Brinch Hansen
None	Working yield A	1.50 1.30	1.10
$H/1,000$	Working yield A	1.25 1.04	

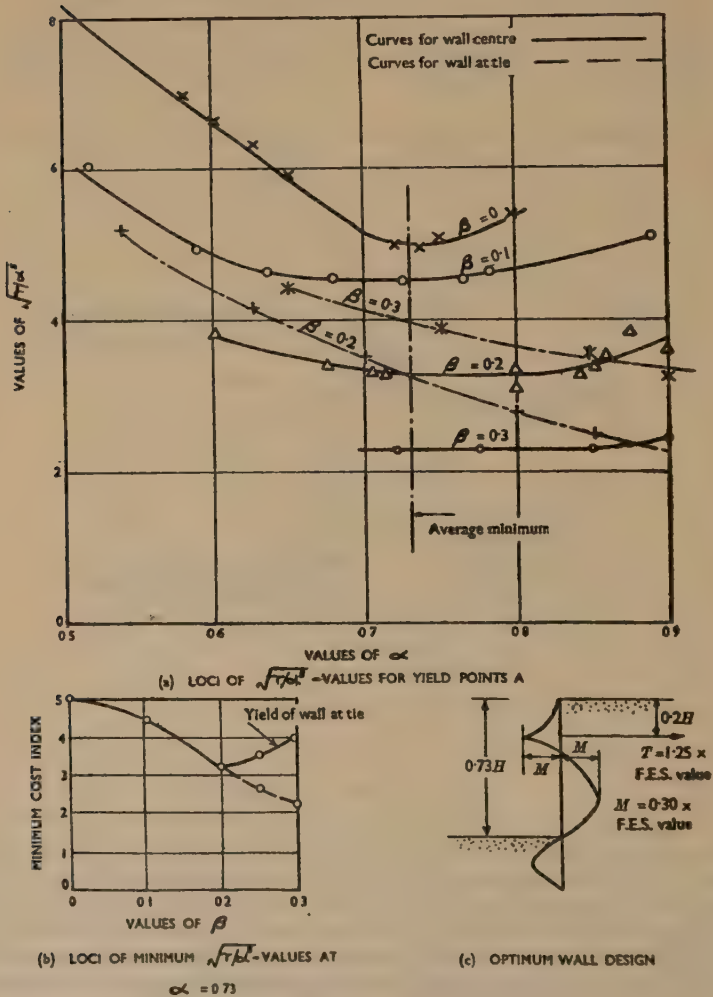


FIG. 30.—EXPERIMENTAL DETERMINATION OF MINIMUM WALL COST

the optimum design by a factor of 1.5. This in part results from the presence of subsidiary causes of moment reduction previously discussed,² but it is partly due to the use of two artificially stiff walls below the dredge level in place of the real continuously deflecting wall. This is equivalent to an arbitrary selection of a passive pressure distribution in a similar manner to the fixed-earth-support method, and indeed the pressure diagram below the dredge level in Fig. 2c is very similar to that used for fixed earth support. The final moment is less than the fixed earth support due to the passive resistance above the ties. However the passive pressure below the dredge level is very sensitive to wall deflexions, and very high pressures can be maintained close to the dredge level and so increase fixity.

The rising moment after yield A means that the factor against ultimate collapse is not very much greater than that against yield A. Therefore the safety factor used by Brinch Hansen against ultimate collapse in conjunction with his bending moment value is of the same order as the actual safety factor against yield A.

Considering now the Author's original work on model walls,¹ the ties were artificially released after each dredge in order to record the maximum bending moments that could possibly act on the wall for normal tie-rod depths. This procedure was also essential in order to study the influence of the relative flexibility of the wall and subsoil. Such yield immediately decreases the tie load, but further deflexion reloads the tie very much quicker than would occur on first loading. As a result, when deflexions occur with tie yield the load is maintained, which confirms Brinch Hansen's and Briske's⁶ opinions regarding the difference between natural and artificial tie yield.

Therefore there are two main causes of moment decrease in anchored walls, namely, (a) decrease with fixity below the dredge level; and (b) decrease with fixity above the tie.

The first of these is the most important in that it is stable and applies with all degrees of tie yield, subsidence, and position of the tie-rods. The second is less certain under field conditions and could only be allowed in design under the understanding that it would cease at yield, of either the wall or the ties.

The tie loads vary as follows: (a) decrease with fixity below the dredge level; and (b) increase with fixity above the tie.

In the result a factor between 1.0 and 1.25 should be applied to the F.E.S. value in the working range, whilst at yield A the factor varies between 0.88 and 1.04. A further factor to allow for differential tie yield seems unnecessary in view of the relation between load and yield.

Now Brinch Hansen's theory is the only one which attempts to take both the above tie-load variations into account, yet under ideal model conditions, the agreement is not substantially better than that of the F.E.S. value. Whatever method is to be used, the tie load calculated will require to be increased in order to allow for such possibilities as settlement of the backfill on to the tie-rods, vibrations, cyclic surcharge loading, or wall creep. The pressure distribution above the tie level is strictly a deformation problem, yet even when this is solved it is highly unlikely that the designer could predict all the values of the parameters required in the solution under field conditions following construction. A practical procedure is therefore to use the F.E.S. tie load as a basis for design, coupled with such load factors as model and, eventually, field measurements dictate.

It is of fundamental importance to study now the reason why relative flexibility of pile and soil does not enter Brinch Hansen's calculation, and also why it appears

not to influence greatly the moments at yield A, apparently confirming the validity of his method.

In Figs 31b to d, three walls are shown penetrating dense sand, loose sand, and a very loose silt respectively. The yield hinge calculations are based on the maximum values of ϕ only, which correspond with strains X, Y, and Z. Therefore the wall deflexions which must occur to mobilize these maximum ϕ values will increase with soil compressibility as illustrated. Yield A will occur at some stage in the deflexion depending on the structural flexibility characteristic of the wall, and the degree of fixity at this stage depends upon the size of the active forces and the compressibility of the subsoil. Thus for loose sand the tests showed that full fixity was reached before yield A, and that thereafter the bending moment was about

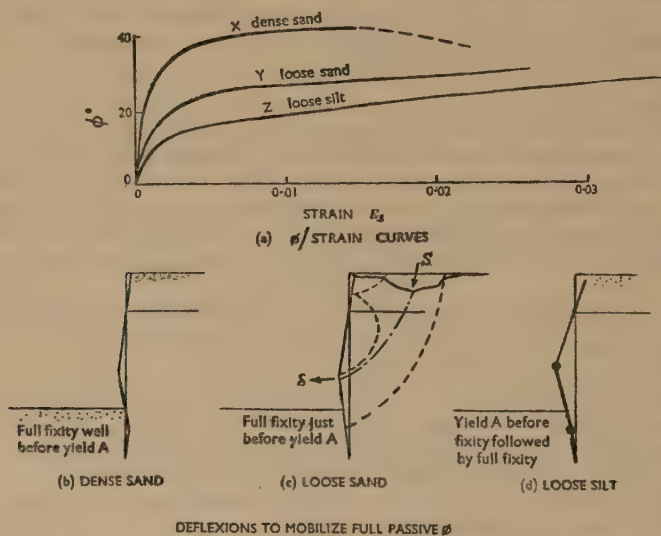


FIG. 31.—INFLUENCE OF SOIL COMPRESSION ON FIXITY AT YIELD

$0.3 \times$ F.E.S. value independent of flexibility. The average value of $\tau_{\max}$ for this case was 10. No sudden stabilizing changes remained to come into force on the development of yield hinges. The same is certain to apply to dense sand for all values of $\tau_{\max}$. Therefore, for nearly all cases in sands, contrary to Brinch Hansen's hypothesis, wall yield is accompanied by increasing moments and, at the ultimate limit, a triangular active pressure distribution applies.

On the other hand for a loose silt subsoil, or for loose sand with $\tau_{\max}$ equal to 16, the wall deflexions at yield A are insufficient to induce full fixity. Neglecting subsidence, wall yield between points A and B would increase flexibility and therefore fixity. The moments would decrease and Brinch Hansen's hypothesis would apply. Owing to subsidence, however, the limit must be yield A, and at this stage the degree of fixity is determined only by consideration of the relative stiffness of the pile and the soil. Thus whilst the yield hinge theory can be applied to clays, excessive deformations may be necessary before the ultimate strength of the soil is approached.

The flexibility method takes the full range of compressibility values for cohesionless

soils into account, with promise of eventual extension to cover clays. Fig. 32 shows two operating curves, one for average field conditions with $\tau_{\max}$ equal to 10 as in the model, and the other with $\tau_{\max}$ equal to 16, the highest value likely to occur. The structural curve for high-strength concrete walls with H equal to 40 ft, which also represents the flexibility characteristic used in the experiments, cuts the lower operating curve at a reduction factor of 0.3. However, concrete piling with ordinary mild-steel reinforcement or sheet steel-piling has a smaller flexure at yield A and cuts the lower operating curve at a factor of 0.38. The corresponding values for the higher operating curve are 0.38 and 0.64 respectively. Finally the operating curves shown apply to loose sand and it has been shown² that these curves would be displaced to the right by $0.75 \log \rho$ units if the piling penetrated a loose silt with $\phi_{\max}$

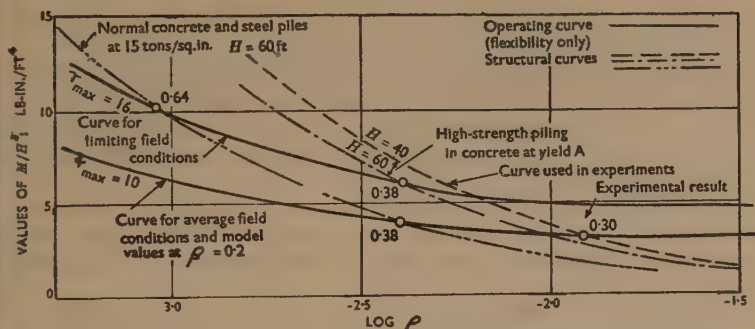


FIG. 32.—INFLUENCE OF FIELD CONDITIONS ON BENDING MOMENTS AT YIELD A

equal to 30° . In that event yield A might occur with no fixity at all. This condition may well apply to normally loaded clay some time after construction.

In using this method it would appear to be reasonable to use the structural curves at yield A or at a limiting deformation rather than those at a safe working stress. In this way the working stress would be nearer yield that is usual in other structures, but the factor of safety would be the same. Thus, if the structural curve at yield gave a moment factor of 0.38 at the intersection of the operating curve, and the steel piling were designed to take this moment at 8 tons/sq. in., then the actual working stress would be 11.5 tons/sq. in. Nevertheless the factor against yield would be 15/9. If this procedure were followed the influence of the different flexure characteristics of the various wall types on the design moment would be much smaller for walls in sand. Further, for sands, it might be considered convenient for design simplification, to tabulate suitable reduction factors on the F.E.S. moment values for particular ranges of sand state and active forces in terms of $\tau_{\max}$.

In conclusion, work is proceeding with other subsoils including clays, and the influences of surcharge and variable strata below the dredge level are being studied. For these reasons the present results should be regarded only as a contribution towards the understanding of the principles involved in sheet-wall design.

It is clear that no design method is likely to take every field condition precisely into account, even supposing that these were all known. Therefore a basic procedure must be adopted with modifications under particular conditions. Naturally, the more general the basic procedure, the fewer are the modifications. This Paper

indicates that either the F.E.S. method with flexibility moment reduction or Brinch Hansen's method may be adopted as the basic procedure. In the former case, a factor may be necessary to the tie load according to the method of construction, position of the ties, surcharge, etc. In the latter case, factors may still be necessary to the tie load for special conditions. In addition, modified ϕ and c values may be necessary to allow for highly compressible soils or cases of high active loading due to water pressure. This would amount simply to the use varying apparent safety factors on the soil constants. The stage is being reached where accurate field measurements are necessary to decide the adoption of safety factors.

CONCLUSIONS

The following conclusions are based on work with loose sand, driven-and-dredged construction, naturally yielding, or encastré anchorage with no surcharge.

(1) The limit to which a sheet-pile wall should be designed is the first yield point. The problem is therefore one of deformation and not of ultimate collapse.

(2) Brinch Hansen's limit method and the Author's flexibility method both provide sound design values in conjunction with first yield for medium field conditions in loose sand subsoil.

(3) Operating and structural curves will always be necessary to determine the bending moment at yield under the full range of field conditions. When designing to the limiting minimum moment for sands they may ultimately be dispensed with under specified conditions.

(4) Active pressure redistribution occurs above the tie level for driven and dredged walls. For loose sand this influence and that of flexibility at wall yield approximately cancel to give the free-earth-support value.

(5) Vertical thrust on the wall influences the form of passive failure of pinned steel walls, and prevents the ultimate collapse of encastré walls.

ACKNOWLEDGEMENTS

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The Author has benefited from the work of Dr J. Brinch Hansen, and from correspondence with him and Dr R. Briske.

REFERENCES

1. P. W. Rowe, "Anchored Sheet-Pile Walls." *Proc. Instn Civ. Engrs*, Pt I, vol. 1, p. 27 (Jan. 1952).
2. P. W. Rowe, "A Theoretical and Experimental Analysis of Sheet-Pile Walls." *Proc. Instn Civ. Engrs*, Pt I, vol. 4, p. 32 (Jan. 1955).
3. J. Brinch Hansen, "Earth Pressure Calculations." The Danish Technical Press, Copenhagen, 1953.
4. P. W. Rowe, "The Flexibility Characteristics of Sheet-Pile Walls." *Struct. Engrg*, vol. 34, p. 150 (May 1955).
5. Correspondence on Reference 1. *Proc. Instn Civ. Engrs*, Pt I, vol. 1, p. 616 (Sept. 1952).
6. R. Briske, "*Erddruckverlagerung bei Spundwandbauwerken*." Wilhelm Ernst u. Sohn, Berlin, 1953.

The Paper, which was received on 18 July, 1955, is accompanied by nine photographs and twenty-four sheets of drawings from which the half-tone page plates and the Figures in the text have been prepared.

CORRESPONDENCE on this Paper should be forwarded to reach the Institution before 1 September, 1956. Contributions should not exceed 1,200 words.—Sec.

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 MACKINLAY, WILLIAM, Grad.I.C.E.
 MAIR, MALCOLM ROBERT, B.A. (Cantab.), Grad.I.C.E.
 MAINDY, GORDON WILLIAM, Grad.I.C.E.
 MARWOOD, HUBERT TREVOR DALRYMPLE, M.A. (Cantab.), Grad.I.C.E.
 MAY, MICHAEL ARTHUR, Grad.I.C.E.
 MEE, WILLIAM THOMAS MICHAEL, B.Sc. (Eng.) (London), Grad.I.C.E.
 MENZIES, IAN WILLIAM, B.Sc. (Glasgow).
 MILLAR, JAMES BRUCE, B.Sc. (Belfast), Grad.I.C.E.
 MILLGATE, LEONARD JOHN, B.Sc.(Eng.) (London), Grad.I.C.E.
 MONTGOMERY, BASIL RICHARD, B.A., B.A.I. (Dublin).
 MOWELL, RONALD STEPHEN, B.Sc.(Eng.) (London), Grad.I.C.E.
 MURRAY, WILLIAM TYRRELL, B.Sc. (Durham), Grad.I.C.E.
 MURCH, HARVEY JENKIN.
 MAGALINGAM, SUPPIRAMANIAM, B.Sc.(Eng.) (London).
 MALL, KENNETH LESLIE HENRY, B.Sc. (Eng.) (London), Grad.I.C.E.
 MATES, RAYMOND KENNETH, Grad.I.C.E.
 MAWLEE, OSWALD JOHN, B.Eng. (Liverpool), Grad.I.C.E.
 MCBORN, DOUGLAS JOHN.
 MCBORNE, ROY GODFREY, B.Sc.(Eng.) (London), Stud.I.C.E.
 MARK, JOHN ANTHONY, B.Sc. (Manchester), Grad.I.C.E.
 MARKER, DEREK RICHARD, B.Sc.(Eng.) (London), Grad.I.C.E.
 MARKER, WILFRED HARRY, B.Sc. (Durham), Grad.I.C.E.
 MATON, DAVID STUART.
 MENDAL, BRYAN JAMES, B.Sc.(Eng.) (London), Grad.I.C.E.
 MERERA, JAMES DENVERS EUSTACE, B.Sc. (Eng.) (London), Grad.I.C.E.
 PHILLIPS, ARNOLD BOYD, M.A. (Cantab.), Grad.I.C.E.
 PINEO, ROGER MICHAEL BRAMLEY, B.Sc. (South Africa), Grad.I.C.E.
 POLLOCK, ALAN GEORGE, B.Sc. (Leeds), Grad.I.C.E.
 POND, ROY VICTOR JAMES, Grad.I.C.E.
 PONRAJAH, ANTHONY JAMES PRINCELY, B.Sc.(Eng.) (London), Grad.I.C.E.
 POTTER, JOHN HORDLEY, B.Sc.Tech. (Manchester), Grad.I.C.E.
 PRENTICE, WILLIAM FRANCIS, B.Sc. (St Andrews), Grad.I.C.E.
 PROSSER, JOHN RUSSELL, B.Sc.(Eng.) (London).
 RAMSAY, DAVID BARCLAY, B.Sc.(Eng.) (London), Grad.I.C.E.
 RANDERSON, MALCOLM DAVID, B.Sc.(Eng.) (London), Grad.I.C.E.
 RATNASINGHAM, THAMPOO RAMALINGAM, B.Sc.(Eng.) (London).
 RAUFF, ABDUL CAFFOOR MOHAMED, B.Sc. (Eng.) (London).
 REAY, CHARLES WILLIAM ANTHONY, B.Sc. (Eng.) (London), Grad.I.C.E.
 REID, THOMAS GORDON, B.Sc. (Leeds), Stud.I.C.E.
 ROBERTS, JOHN FAWCETT, B.Sc. (Leeds), Grad.I.C.E.
 RODGER, WILLIAM GIBSON, B.Sc. (Glasgow), Grad.I.C.E.
 ROONEY, JAMES DESMOND, B.Sc.(Eng.) (London), Grad.I.C.E.
 SAMUEL, RONALD HERBERT, B.Eng. (Liverpool), Grad.I.C.E.
 SAVUNDRANAYAGAM, EMMANUEL TIRU-
 CHELVAM, B.Sc.(Eng.) (London), Grad.
 I.C.E.
 SENEVIRATNE, JOSEPH RIENZI VISTRINE, B.Sc.(Eng.) (London), Grad.I.C.E.
 SIVALOGANATHAN, KATHIRITHAMBY, B.Sc. (Eng.) (London), Grad.I.C.E.
 SMITH, LESLIE DEREK, B.Sc.(Eng.) (London), Grad.I.C.E.
 SODEN, JOHN KENNETH, B.Sc. (Belfast), Grad.I.C.E.
 STREETEN, BASIL RICHARD, B.A. (Cantab.), Grad.I.C.E.
 SURRIDGE, KENNETH JAMES, B.Sc.(Eng.) (London), Grad.I.C.E.
 TAYLOR, WILLIAM ALEXANDER, B.Sc. (Aberdeen).
 THOMPSON, JAMES LAURENCE, B.Sc.(Eng.) (London), Grad.I.C.E.
 THOMSON, ANGUS JAMES ORR, B.A., (Cantab.), Grad.I.C.E.
 TURNER, JAMES KENNETH, B.Sc. (Wales), Grad.I.C.E.
 VAN HEERDEN, ANTONIE, B.Sc. (Cape Town), Grad.I.C.E.
 VEERAKATHY, SIVAKOLUNDU, B.Sc. (Madras).
 VITALI, LOUIS JAMES, B.Sc.(Eng.) (London), Grad.I.C.E.

WALKER, WILLIAM ROBIN, B.Sc.(Eng.) (<i>London</i>), Grad.I.C.E.	WEIR, JOHN MALCOLM TWEEDDALE, Grad. I.C.E.
WARD, MICHAEL HENRY.	WILKINSON, ALAN RICHARD.
WATSON, CHRISTOPHER HUGH, Grad.I.C.E.	WILSON, FRED, B.Sc.(Eng.) (<i>London</i>), Grad. I.C.E.
WEERARATNE, DON LAMBERT JUSTIN, B.Sc.(Eng.) (<i>London</i>).	WELLS, ALAN PAUL, B.Sc.Tech. (<i>Manchester</i>), Grad.I.C.E.

Associates

DIXON, JOHN CHAPMAN, B.Sc.

UNDERHILL, HAROLD WADE, B.A. (*Cantab.*).

DEATHS

It is with deep regret that intimation of the deaths of the following has been received.

Members

- ALFRED HENRY ASLETT (E. 1892, T. 1920).
 WILLIAM EWART BLIZARD, B.Sc.(Eng.) (E. 1916, T. 1929).
 THEOBALD STUART BUTLER (E. 1910, T. 1926).
 PHILIP GROSVENOR CORIN, M.A. (E. 1932, T. 1936).
 EDWARD THOMAS FREDERICK ELBURY, B.Sc. (E. 1935, T. 1949).
 GORDON RATTRAY FENTON, O.B.E. (E. 1926, T. 1942).
 JOHN WISHART FAIRLIE GARDNER (E. 1894, T. 1905).
 PHILIP HAMMOND (E. 1891, T. 1920).
 ROBERT WILLIAM LESLIE HARRIS, B.Sc. (E. 1942, T. 1945).
 JOHN MACKINTOSH HOGG (E. 1909, T. 1919).
 RICHARD JOSEPH HOWLEY, C.B.E. (E. 1897, T. 1912).
 Sir JOHN ROBERT KEMP, M.E. (E. 1929).
 FREDERICK ROBERT MELVILLE, B.Sc. (E. 1912, T. 1930).
 DONALD MOIR (E. 1935).
 HENRY KNIGHT RODGERS (E. 1929, T. 1939).
 PERCY STABKEY SPENCER (E. 1911, T. 1929).
 GEORGE STOW, O.B.E. (E. 1901, T. 1953).
 PHILIP HOWARD THORNE, M.C. (E. 1920, T. 1928).
 JAMES CECIL LIONEL VERLEY (E. 1905, T. 1920).
 DONALD MURRAY WALKER, M.Eng. (E. 1929, T. 1950).
 HAROLD WHIPP, O.B.E., B.Sc. (E. 1928, T. 1948).

Associate Members

- CHARLES GODFREY BOYTON, M.A. (E. 1944).
 JAMES BURDON (E. 1939).
 JAMES DENYS BUSH, B.Sc. (E. 1925).
 JAMES CRUIKSHANK (E. 1925).
 JOHN BERTRAM DONAT, B.Sc.(Eng.) (E. 1950).
 GERALD STEPHEN DUNKIN (E. 1925).
 HENRY WALTER FORD (E. 1883).
 JOHN RICHARD STRATFORD FOX (E. 1893).
 DAVID JOHNSON GADSBY (E. 1900).
 WILLIAM MACLEAN HOMAN (E. 1896).
 FRANCIS BOVILL HIGGINS (E. 1896).
 RONALD ALEXANDER INGLIS, B.Sc.(Eng.) (E. 1915).
 EDWYN JERVOISE (E. 1911).
 LEONARD LACEY, B.Sc.(Eng.) (E. 1920).
 ALEXANDER SIMPSON MACDONALD, B.Sc. (E. 1921).
 THOMAS ALBERT NEILL (E. 1904).
 WALTER REEVE (E. 1906).
 STANLEY LEWIS RICHARDS (E. 1918).
 PERCY FRANCIS SPILLER (E. 1916).
 WALTER PERCHARD STERICKER (E. 1897).
 DAVID BOWE WATERS, B.Sc. (E. 1939).
 PERCY WILKINSON (E. 1909).

Graduate

- THANUVELIL KOSHY GEORGE, M.Sc.Tech. (A. 1952).

OBITUARY

GRAHAM TOWNSEND BENNETT, O.B.E., B.Sc., who was born on 30 October, 1895, died on 28 November, 1955.

He was educated at the Merchant Venturers' School, Bristol, and graduated at the University of Bristol. He completed his studies in 1919, after commissioned service in the Worcester Regiment during the 1914-18 War. From 1920 to 1925, he was apprenticed as a Junior Assistant to Messrs A. P. Cotterell & Sons, consulting engineers in waterworks and sewerage. From 1925 to 1927 he was an Engineering Assistant with the Oxfordshire County Council.

In 1927, he was appointed County Surveyor of Oxfordshire, and in 1948 County Surveyor of Buckinghamshire. From 1940 to 1945, he served with the Royal Engineers in the Middle East and later in France and Germany. He was Mentioned in Dispatches four times and was awarded the Order of the British Empire (Military).

He was a leading authority on highways and was particularly concerned with the relation of road layout to traffic accidents. He achieved considerable success in reducing fatal road accidents in Oxfordshire by alterations to a number of road junctions.

Mr Bennett was the Author of many Papers on road and bridge problems and presented a Paper¹ to the Joint Conference of the Institutions of Civil, of Mechanical, and of Electrical Engineers. In April 1949, in joint authorship with R. J. Smeed, he presented to the Road Engineering Division a Paper on "Research on Traffic Flow and Road Safety".²

He was elected an Associate Member in 1922, and transferred to the class of Member in 1945. He was a Member of Council from 1951 until the end of October 1955. He then declined to stand for re-election on account of his failing health.

He was also a Member of the Institution of Municipal Engineers, a Fellow and Past-President of the Institution of Highway Engineers, and a Past-President (and later Secretary) of the County Surveyors' Society.

He is survived by his wife, two sons, and a daughter.

ERNEST HONE FORD, O.B.E., who was born on 11 August, 1884, died on 15 November, 1955.

From 1902 to 1905 he was articled to the Borough Engineer of Walsall, and was then Assistant to the Borough Surveyor at Walsall until 1912, when he was appointed Assistant Surveyor, Southport. In 1924 he became City Engineer and Surveyor, and Planning Officer, to Coventry Corporation, which office he held until his retirement in 1949. He remained in private practice as a consulting engineer and Town and Country Planning consultant until July 1954, and was Chairman of the Coventry Cathedral Reconstruction Committee until his death.

His Paper on "Coventry By-Pass Road" read before a Joint Meeting of the Birmingham and District Association of the Institution and the West Midland

¹ "Road Planning and Safety". Proc. Jt Engng Conf., I.C.E., I.Mech.E., I.E.E., 1931, p. 229.

² Road Paper 29, Instn Civ. Engrs, 1949.

District Branch of the Institution of Municipal and County Engineers in 1940 was published in the Journal¹ of the latter Society and, in an abridged form, in the Institution Journal.²

During the 1939-45 War, he was actively concerned with the numerous road, housing, and sewerage schemes, and the air raid precautions necessitated by the importance of Coventry as an industrial centre.

He was elected an Associate Member in 1911, and was transferred to the class of Member in 1931. He was also a Past-President of the Institution of Municipal and County Engineers, Past-President of the Town Planning Institute, a Fellow of the Royal Sanitary Institute, and a Fellow of the Royal Society of Arts.

He is survived by his wife and two daughters and a son, who is also a civil engineer.

ARTHUR GRAHAM GLASGOW, M.E., who died on 28 October, 1955, was born in the United States of America on 30 May, 1865.

He graduated in Engineering at Stevens Institute of Technology, in 1885, and subsequently had conferred on him, in 1928, the Honorary Degree of Doctor of Engineering. He was made an Honorary Doctor of Science of Washington and Lee University, in 1930, and an Honorary Doctor of Law of Wabash College, 1950.

He was an Honorary Member of the Institution of Gas Engineers, a Member of the Institution of Mechanical Engineers, of the American Society of Civil Engineers, and of the American Society of Mechanical Engineers. He was a Gold Medallist of the American Gas Association, and of the Franklin Institute of Pennsylvania.

Dr Glasgow's principal interest lay in gas production and processes, and to this end he established the firm of Humphreys and Glasgow Ltd in 1892, and remained as Chairman until 1952, but retained an interest in the firm until his death. He was also Chairman and President of the Building Supplies Corporation of Norfolk, Virginia.

He was elected an Associate Member in 1898, and was transferred to the class of Members in the same year.

ARTHUR STANLEY GLOVER, M.C., B.Sc., who was born on 24 April, 1889, died on 12 November, 1955.

He graduated from Durham University in 1912.

He held an appointment as Engineering Assistant with the Roads Department, Ministry of Transport, and from 1922 to 1931 was Assistant County Surveyor, to the Durham County Council. From 1931 until his death, he was County Surveyor and Bridge Master of the County of Westmorland.

During the 1914-18 War he served in the Royal Engineers, attaining the rank of Captain. He was awarded the Military Cross and Croix de Guerre.

He was elected Associate Member in 1918, and was also a Member of the Institution of Municipal Engineers.

He is survived by his wife, Mrs Doris Glover.

JOHN MACKINTOSH HOGG, who was born on 11 November, 1882, died at Glasgow on 22 December, 1955.

He served his apprenticeship in the firm of Messrs Crouch & Hogg, of which his father, C. P. Hogg, had been one of the founders. He later became Assistant Engineer and Resident Engineer engaged on the firm's reservoir works and other

¹ J. Instn M. & Cty Engrs, vol. 66, p. 596 (Feb. 1940).

² J. Instn Civ. Engrs, vol. 14, p. 238 (Apr. 1940).

projects. He became a Partner of the firm in 1910, and senior Partner in 1927 on the retirement of his father.

He was especially engaged in the planning and construction of waterworks for a number of Local Authorities in Scotland and more recently he was concerned with regional water schemes for the County Councils of Angus, Kincardine, Orkney, Roxburgh, and Wigtown. He was also concerned with important works for the North of Scotland Hydro-Electric Board, including the Cowal Scheme, and with various main drainage schemes and other works.

In 1925, the Crown Authorities appointed him to report on the Skelmorlie (Ayrshire) Reservoir disaster and he was well known as an Arbitrator and Expert Witness.

He was an original Member of the Panel of Qualified Civil Engineers set up under the Reservoirs (Safety Provisions) Act, 1930.

He was elected an Associate Member in 1909 and was transferred to the class of Member in 1919.

He was a Past-Chairman of the Glasgow and West of Scotland Association and a Past-President of the Scottish section of the Institution of Water Engineers.

He is survived by three sons.

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